SIGNAL AND SYSTEM (BTECH 4TH SEM) EE

UNIT – 1

Signal and system:- Signal is a function which contains some information or message. Example- x(t)=2y(t)

And system is a interconnection of components or devices.

Signal properties:-

Periodicity:- A signal is a periodic signal if it completes a pattern within a measurable time frame, called a period and repeats that pattern over identical subsequent periods. The completion of a full pattern is called a cycle. A period is defined as the amount of time (expressed in seconds) required to complete one full cycle. The duration of a period represented by T, may be different for each signal but it is constant for any given periodic signal.

The **period** is the smallest value of T satisfying $g(t + T) = g(t)$ for all t. The period is defined so because if $g(t + T) = g(t)$ for all t, it can be verified that $g(t + T') = g(t)$ for all t where $T' = 2T, 3T, 4T, ...$ In essence, it's the smallest amount of time it takes for the function to repeat itself. If the period of a function is finite, the function is called "periodic". Functions that never repeat themselves have an infinite period, and are known as "aperiodic functions".

The period of a periodic waveform will be denoted with a capital T. The period is measured in seconds.

Absolute integrability:-

An **absolutely integrable function** is a function whose absolute value is integrable, meaning that the integral of the absolute value over the whole domain is finite.

For a real-valued function, since

Consider a measure space (X,A,μ)(X,A,μ). A measurable function $f:X\to[-\infty,\infty]$f:X→[−∞,∞] is then called absolutely integrable if

$\int|f|d\mu<\infty$.

An absolutely integrable function is also commonly called a *summable function*.

Remark If we assume only the measurability of $|f|$|f|, then this does not guarantee the measurability of ff. Although a few authors require only the measurability of $|f|$|f|, the vast majority of the literature assumes that ff itself is measurable.

The following inequality, which is a particular case of Jensen's inequality, holds for any absolutely integrable function:

$|||\int f d\mu||| \leq \int |f| d\mu$

(the assumption of absolute integrability is however not fundamental: the inequality makes sense and holds as soon as we can define

$\int f d\mu$,

that is, as soon as the integral of the positive part of ff or that of the negative part of ff are finite).

The space of absolutely integrable functions is a linear space which is usually denoted by $L_1(X,\mu)$L1(X,μ) and

$\|f\|_1 := \int |f| d\mu < \infty$

is a seminorm on it. It is customary to identify elements of L1(X,μ)L1(X,μ) whose values coincide except for a μμ-null set: after this identification the norme ‖⋅‖1‖⋅‖1 endowes L1(X,μ)L1(X,μ) with a Banach space structure. The L1L1 space is then just one case of a more general class of Banach spaces called Lp spaces.

Determinism:-

A signal is classified as *deterministic* if it's a completely specified function of time. A good example of a deterministic signal is a signal composed of a single sinusoid, such as

$$x(t) = A\cos(2\pi f_0 t + \phi)$$

with the signal parameters being:

$$(A, f_0, \phi)$$

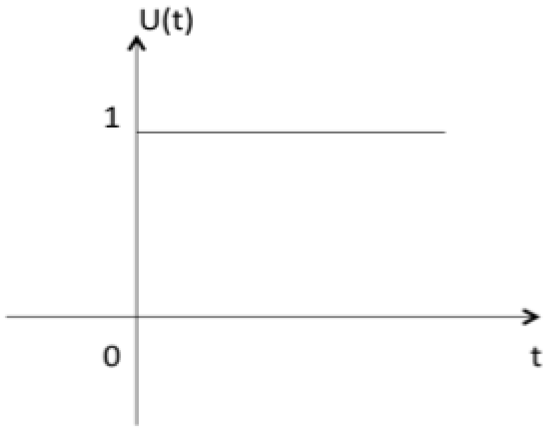
A is the amplitude, f_0 is the frequency (oscillation rate) in cycles per second (or hertz), and ϕ

is the phase in radians. Depending on your background, you may be more familiar with radian frequency,

$$\omega_0 = 2\pi f_0$$

which has units of radians/sample. In any case, *x*(*t*) is deterministic because the signal parameters are constants.

Unit Step signal:-

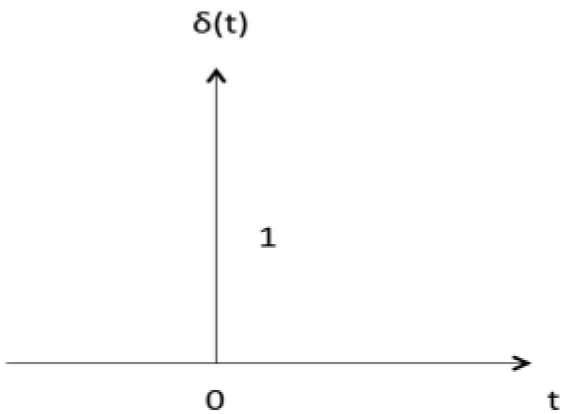


Unit step function is denoted by u(t). It is defined as u(t) = $\begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$

It is used as best test signal.

Area under unit step function is unity.

<u>Unit Impulse signal</u>:-

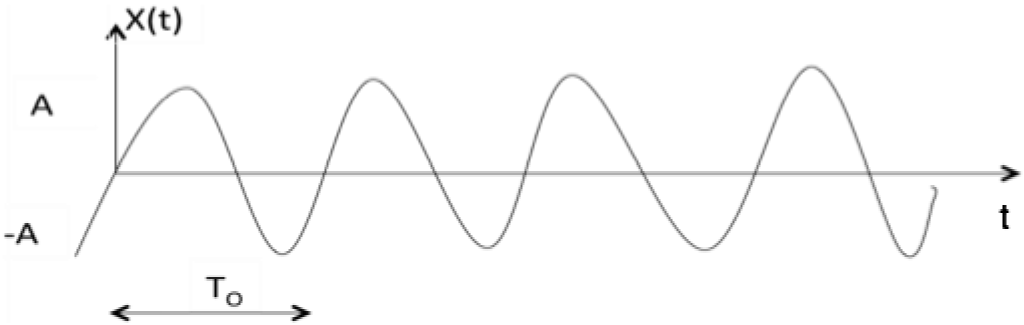


Impulse function is denoted by δ(t). and it is defined as δ(t) = $\begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

<u>Sinusoidal Signals</u>:-



Sinusoidal signal is in the form of x(t) = A cos($w_0 \pm \phi$w0±ϕ) or A sin($w_0 \pm \phi$w0±ϕ)

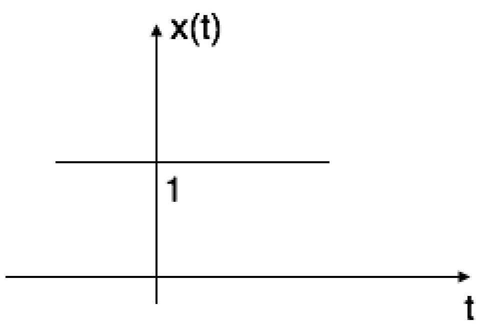
Where $T_0 = 2\pi/w0$

<u>Exponential Signals</u>:-

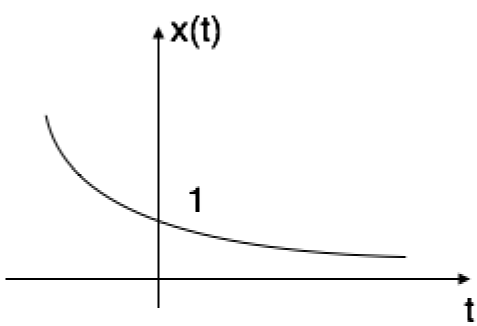
Exponential signal is in the form of x(t) = $e^{\alpha t}$

The shape of exponential can be defined by αα

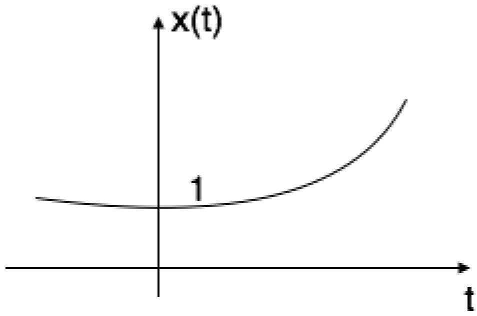
Case i: if αα = 0 →→ x(t) = e^0e0 = 1



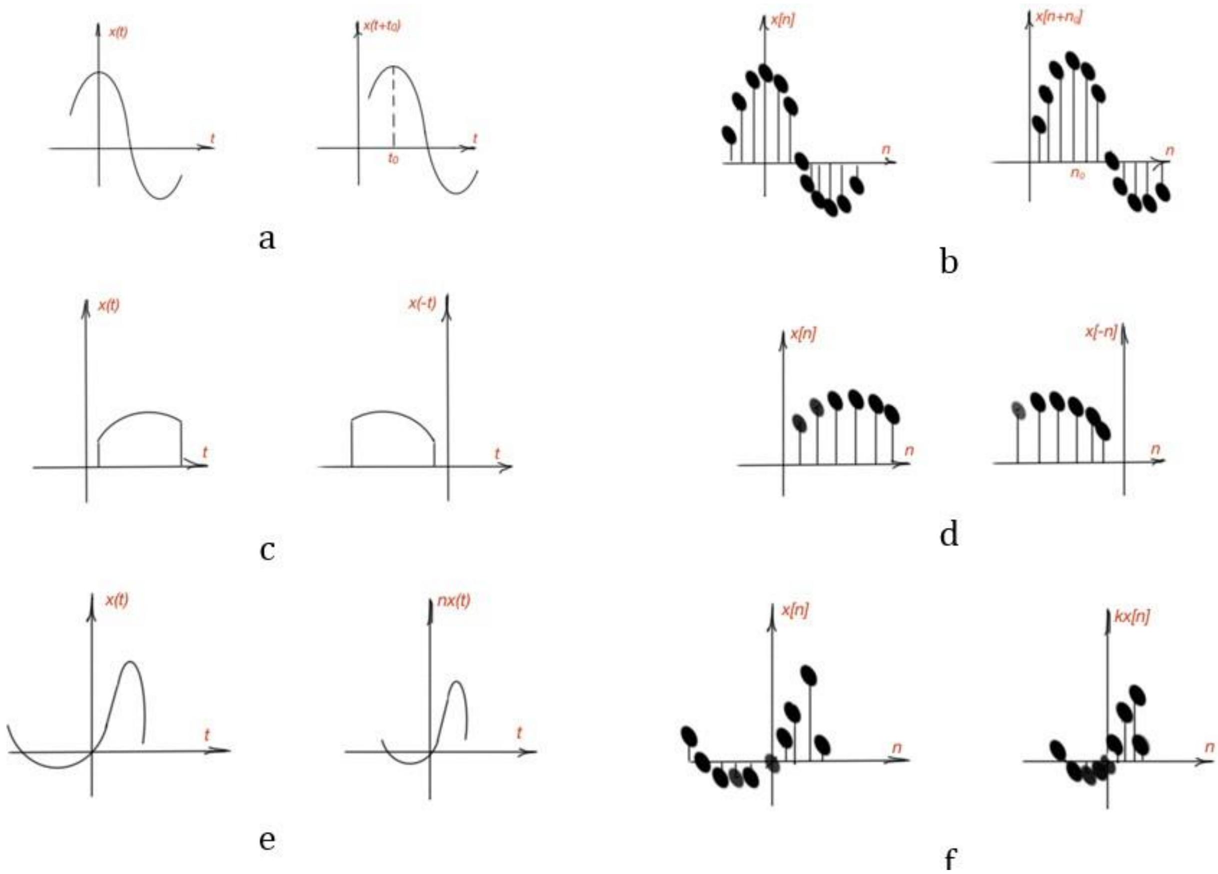
Case ii: if αα < 0 i.e. -ve then x(t) = $e^{-\alpha t}$e−αt. The shape is called decaying exponential.



Case iii: if αα > 0 i.e. +ve then x(t) = $e^{\alpha t}$eαt. The shape is called raising exponential.



<u>Discrete time and Continous time signals</u>:-

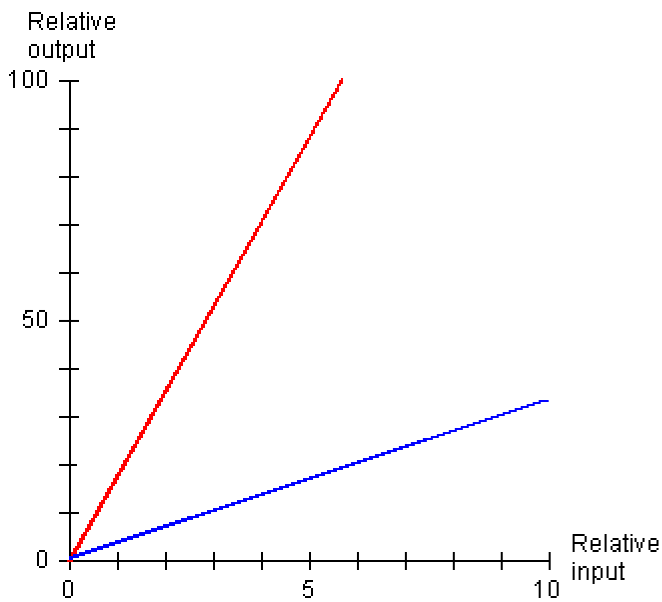


Continuous-time signals are characterised by independent variables that are continuous and define a continuous set of values. Usually the variable indicates the continuous time signals, and the variable *n* indicates the discrete-time system. Also the independent variable is enclosed at parentheses for continuous-time signals and to the brackets for discrete-time systems. The feature of the discrete-time signals is that they are sampling continuous-time signals.

<u>Linearity</u>:-

Linearity is the behavior of a circuit, particularly an amplifier , in which the output signal strength varies in direct proportion to the input signal strength. In a linear device, the output-to-input signal amplitude ratio is always the same, no matter what the strength of the input signal (as long it is not too strong).

In an amplifier that exhibits linearity, the output-versus-input signal amplitude graph appears as a straight line. Two examples are shown below. The gain, or amplification factor, determines the slope of the line. The steeper the slope, the greater the gain. The amplifier depicted by the red line has more gain than the one depicted by the blue line. Both amplifiers are linear within the input-signal strength range shown, because both lines in the graph are straight.



Examples of linearity

In analog applications such as amplitude-modulation (AM) wireless transmission and hi-fi audio, linearity is important. Nonlinearity in these applications results in signal distortion, because the fluctuation in gain affects the shape of an analog output waveform with respect to the analog input waveform.

Even if an amplifier exhibits linearity under normal conditions, it will become nonlinear if the input signal is too strong. This situation is called overdrive. The amplification curve bends towards a horizontal slope as the input-signal amplitude increases beyond the critical point, producing distortion in the output. An example is a hi-fi amplifier whose gain is set to the point where the VU (volume-unit) meter needles kick into the red range. The red zone indicates that the amplifier is not operating in a linear fashion. This can degrade the fidelity of the sound.

Shift invariance:-

A linear differential equation with constant coefficients displays time invariance. If we use the same input and starting conditions for a system now or at some later time then the result relative to the initial starting time will be identical. Another way of expressing this is that if the input is time shifted then so is the output.

Differential equations and difference equations

Mary Attenborough, in Mathematics for Electrical Engineering and Computing, 2003

Time invariance

A linear differential equation with constant coefficients displays time invariance. If we use the same input and starting conditions for a system now or at some later time then the result relative to the initial starting time will be identical. Another way of expressing this is that if the input is time shifted then so is the output. This idea is represented in Figure 14.5.

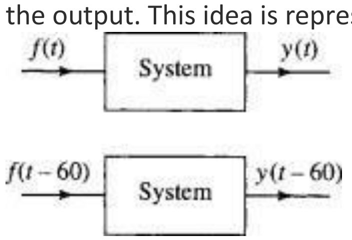


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Figure 14.5. If a system is time invariant, then a time-shifted input yields a time-shifted output.

Example 14.4

For the differential equation

$$d^2y/dt^2+4y=\sin(3t)$$

show that

$$y=\sin(2t)-15\sin(3t)$$

is a solution and find a solution for the equation with the same input function delayed by 1 s, that is, find a solution to

$$d^2y/dt^2+4y=\sin(3(t-1))$$

Solution First, we check that

$$y=\sin(2t)-15\sin(3t)$$

is a solution to the differential equation.

To do this, we must find the first and second derivatives

$$dy/dt=2\cos(2t)-(3/5)\cos(3t) \quad d^2y/dt^2=-4\sin(2t)+(9/5)\sin(3t)$$

Substitute into

$$d^2y/dt^2+4y=\sin(3t)$$

giving

$$-4\sin(2t)+(9/5)\sin(3t)+4(\sin(2t)-15\sin(3t))=\sin(3t) \quad \Leftrightarrow \quad \sin(3t)=\sin(3t)$$

which is true for all *t.* Hence,
y=sin(2t)−15sin(3t)

is a solution to

d2y/dt2+4y=sin(3t).

To find a solution to $d^2y/dt^2 + 4y = \sin(3(t - 1))$, we use the property of time invariance, which means that a solution should be given by a time-shifted version of the solution to the first equation, that is
y=sin(2(t−1))−15sin(3(t−1))

then

dy/dt=2cos(2(t−1))−(3/5)cos(3(t−1))d2y/dt2=−4sin(2(t−1))+(9/5)sin(3(t−1)).

Substitute into

d2y/dt2+4y=sin(3(t−1))

giving

−4sin(2(t−1))+(9/5)sin(3(t−1))+4(sin(2(t−1))−(1/5)sin(3(t−1)))=sin(3(t−1))⇔ sin(3(t−1))=sin(3(t−1))

which is true for all *t.*
Example 14.5

For the differential equation

t dy/dt+y=6t2

(a)

show that $y = 2t^2$ is a solution

(b)

show that the equation $t\, dy/dt + y = 6t^2$ cannot represent a time invariant system.

Solution (a) To show that $y = 2t^2$ is a solution to
t dy/dt+y=6t2

we need to find dy/dt
y=2t2 ⇒dy/dt=4t

substituting into

t dy/dt+y=6t2

gives

t (4t)+2t2=6t2 ⇔ 6t2=6t2, which is true for all t.

(b) To show that this cannot represent a time-invariant system, we take an equation with a time-shifted input, for instance, shifted by 2 s to give

t dy/dt+y=6(t−2)2.

If it were to be time-invariant, then a solution to this equation would be a time-shifted solution of the solution to the equation in part (a), that is, $y = 2(t - 2)^2$. To show that the equation does not represent a time-invariant system we just need to show that $y = 2(t - 2)^2$ is not a solution

y=2(t−2)2 ⇒dy/dt=4(t−2)

Substitution into

t dy/dt+y=6(t−2)2

gives

t(4(t−2))+2(t−2)2=6(t−2)2 ⇔ 4t(t−2)+2(t−2)2−6(t−2)2=0 ⇔ (t−2)(4t+2t(t−2)−6(t−2))=0 ⇔ (t−2)(4t+2t−4−6t+12)=0 ⇔ 8(t−2)=0

which is not true for all values of t, showing that $y = 2(t - 2)^2$ is not a solution to $t\,dy/dt + y = 6(t - 2)^2$ and therefore we have shown that $t\,dy/dt + y = f(t)$ does not represent a time-invariant system.

We have seen that linear differential equations with constant coefficients represent linear time invariant (LTI) systems.

Causality:- A **causal** system is one whose output depends only on the present and the past inputs.

A noncausal system's output depends on the future inputs. In a sense, a noncausal system is just the opposite of one that has memory.

A causal system is the one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, and its past input sample values such as $x(n - 1)$, $x(n - 2)$,…. Otherwise, if a system output depends on the future input values such as $x(n + 1)$, $x(n + 2)$,…, the system is noncausal. The noncausal system cannot be realized in real time.

Stability:-

BIBO bounded input forboundedoutputboundedinputforboundedoutput condition.Here, bounded means finite in amplitude. For a stable system, output should be bounded or finite, for finite or bounded input, at every instant of time.

Some examples of bounded inputs are functions of sine, cosine, DC, signum and unit step.

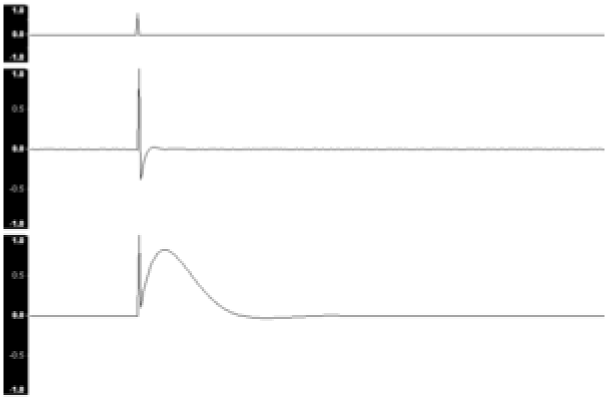
Realizability:-

Realizability. In mathematical logic, **realizability** is a collection of methods in proof theory used to study constructive proofs and extract additional information from them. Most variants of **realizability** begin with a theorem that any statement that is provable in the formal **system** being studied is realizable.

UNIT-2

Impulse Response:-

In signal processing, the **impulse response**, or **impulse response function (IRF)**, of a dynamic system is its output when presented with a brief input signal, called an impulse. More generally, an impulse response is the reaction of any dynamic system in response to some external change. In both cases, the impulse response describes the reaction of the system as a function of time (or possibly as a function of some other independent variable that parameterizes the dynamic behavior of the system).

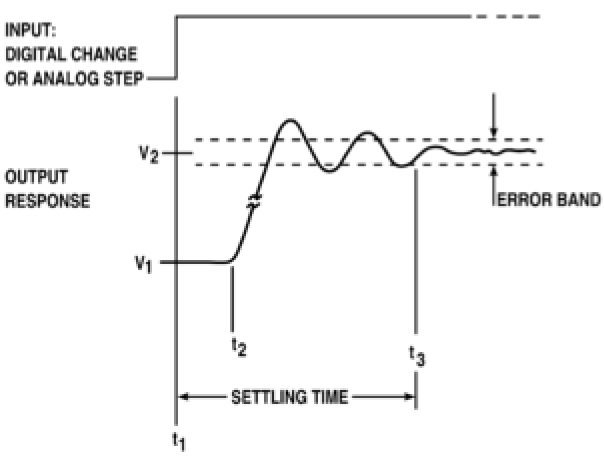


In all these cases, the dynamic system and its impulse response may be actual physical objects, or may be mathematical systems of equations describing such objects.

Since the impulse function contains all frequencies, the impulse response defines the response of a linear time-invariant system for all frequencies.

Step Response:-

The **step response** of a system in a given initial state consists of the time evolution of its outputs when its control inputs are Heaviside step functions. In electronic engineering and control theory, step response is the time behaviour of the outputs of a general system when its inputs change from zero to one in a very short time. The concept can be extended to the abstract mathematical notion of a dynamical system using an evolution parameter.



From a practical standpoint, knowing how the system responds to a sudden input is important because large and possibly fast deviations from the long term steady state may have extreme effects on the component itself and on other portions of the overall system dependent on this component. In addition, the overall system cannot act until the component's output settles down to some vicinity of its final state, delaying the overall system response. Formally, knowing the step response of a dynamical system gives information on the stability of such a system, and on its ability to reach one stationary state when starting from another.

Convolution:-

The *convolution theorem* offers an elegant alternative to finding the inverse Laplace transform of a function that can be written as the product of two functions, without using the simple fraction expansion process, which, at times, could be quite complex, as we see later in this chapter. The convolution theorem is based on the *convolution of two functions f(t)* and *g(t)*. According to the definition, the convolution of *f(t)* and *g(t)* is

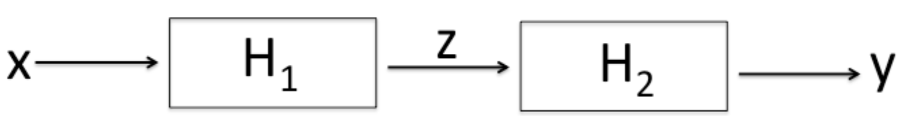
$$
\begin{aligned}
f_1(t) * f_2(t) &= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau \\
&= \int_0^t f_2(\tau) f_1(t-\tau)\, d\tau \qquad \ldots (1)
\end{aligned}
$$

It is straightforward to demonstrate that the convolution of two functions is a commutative operation.

Cascade interconnections:-

Systems can be combined to form more complex systems otherwise known as the **interconnection of systems**. We will generically use xx and yy without x(t)x(t) and x[n]x[n] in this discussion with the understanding that this notation applies to both CT and DT systems.

A series or casade interconnection is the results of an input xx into system H1H1 which results in an output zz that is in turn the input for system H2H2 which results in an output yy as illustrated below.

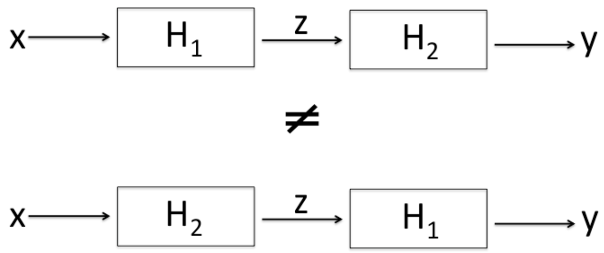


A cascade system can mathematically be represented as:

$$z=H_1\cdot xz=H1\cdot x$$

$$y=H_2\cdot zy=H2\cdot z$$

$$y=H_1\cdot H_1\cdot x$$



Case 1: $y=H_2\cdot H_1\cdot x$Case 1: y=H2·H1·x

Case2: $y=H_1\cdot H_2\cdot x$Case2: y=H1·H2·x

In general
$H_1\cdot H_2\neq H_2\cdot H_1$H1·H2≠H2·H1

exept for some special cases

Multi input Multi Output representation of systems:-

Systems with more than one input and/or more than one output are known as **Multi-Input Multi-Output** systems, or they are frequently known by the abbreviation **MIMO**. This is in contrast to systems that have only a single input and a single output (SISO).

MIMO systems that are lumped and linear can be described easily with state-space equations. To represent multiple inputs we expand the input *u(t)* into a vector *U*(t) with the desired number of inputs. Likewise, to represent a system with multiple outputs, we expand *y(t)* into *Y*(t), which is a vector of all the outputs. For this method to work, the outputs must be linearly dependent on the input vector and the state vector.

X’((t) = A X(t) + B U(t)

Y(t) = C X(t) + D U(t)

STATE VARIABLE REPRESENTATIONS BY VARIOUS METHODS

The State Variables of a Dynamic System

The State Differential Equation

Signal-Flow Graph State Variables

The Transfer Function from the State Equation

Introduction

- In the previous chapter, we used Laplace transform to obtain the transfer function models representing linear, time-invariant, physical systems utilizing block diagrams to interconnect systems.

- In Chapter 3, we turn to an alternative method of system modeling using **time-domain methods**.

- In Chapter 3, we will consider physical systems described by an **nth-order ordinary differential equations**.

- Utilizing a set of variables known as **state variables**, we can obtain a set of first-order differential equations.

- The time-domain state variable model lends itself easily to computer solution and analysis.

Time-Varying Control System

- With the ready availability of digital computers, it is convenient to consider the time-domain formulation of the equations representing control systems.

- The time-domain is the mathematical domain that incorporates the response and description of a system in terms of time t.

- The time-domain techniques can be utilized for nonlinear, time-varying, and multivariable systems (a system with several input and output signals).

- A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time.

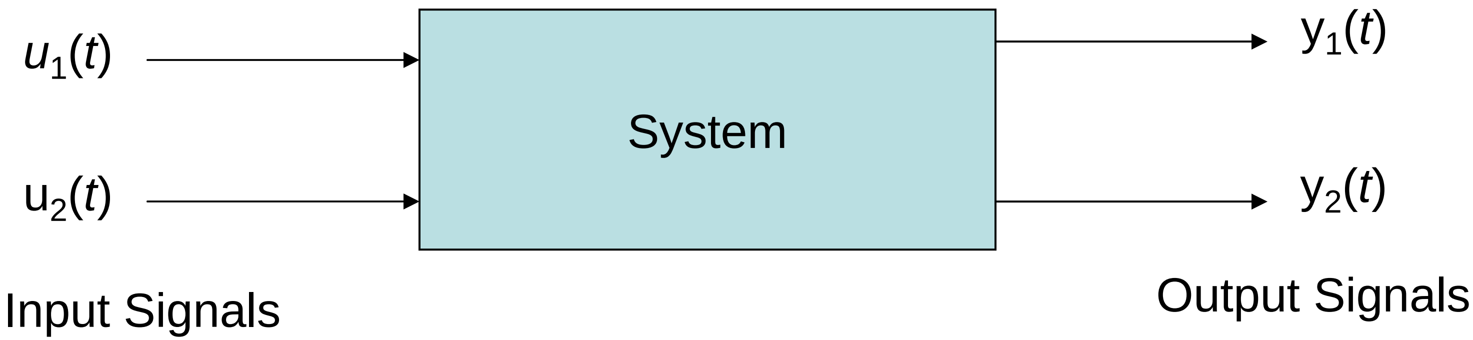
- For example, the mass of a missile varies as a function of time as the fuel is expended during flight

Terms

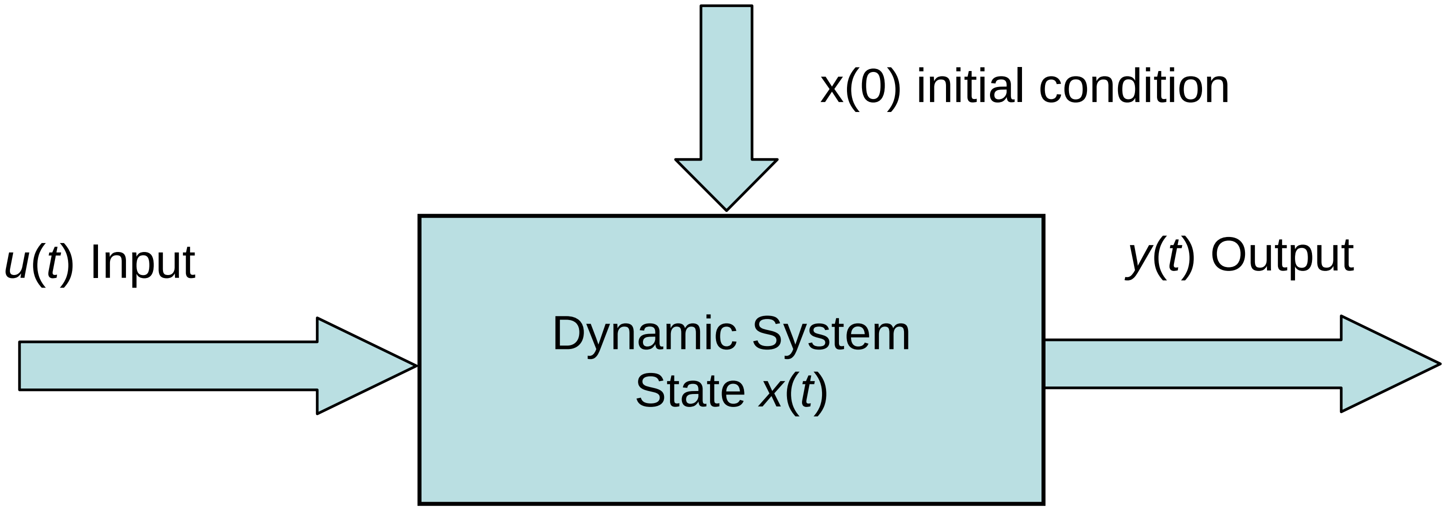
- **State:** The state of a dynamic system is the smallest set of variables (called state variables) so that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, determines the behavior of the system for any time $t \geq t_0$.
- **State Variables:** The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.
- **State Vector:** If n state variables are needed to describe the behavior of a given system, then the n state variables can be considered the n components of a vector x. Such vector is called a state vector.
- **State Space:** The n-dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, .., x_n axis, where x_1, x_2, .., x_n are state variables, is called a state space.
- **State-Space Equations:** In state-space analysis, we are concerned with three types of variables that are involved in the modeling of dynamic system: input variables, output variables, and state variables.

The State Variables of a Dynamic System

- The state of a system is a set of variables such that the knowledge of these variables and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.
- For a dynamic system, the state of a system is described in terms of a set of state variables.



State Variables of a Dynamic System



The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics

The State Differential Equation

The state of a system is described by the set of first-order differential equations written in terms of the state variables (x_1, x_2, .., x_n)

$$\dot{x} = \frac{dx}{dt}$$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{x}} + \underbrace{\begin{bmatrix} b_{11} & \ldots & b_{1m} \\ & \ldots\ldots & \\ b_{n1} & \ldots & b_{nm} \end{bmatrix}}_{\text{B}} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}}_{\text{u}}$$

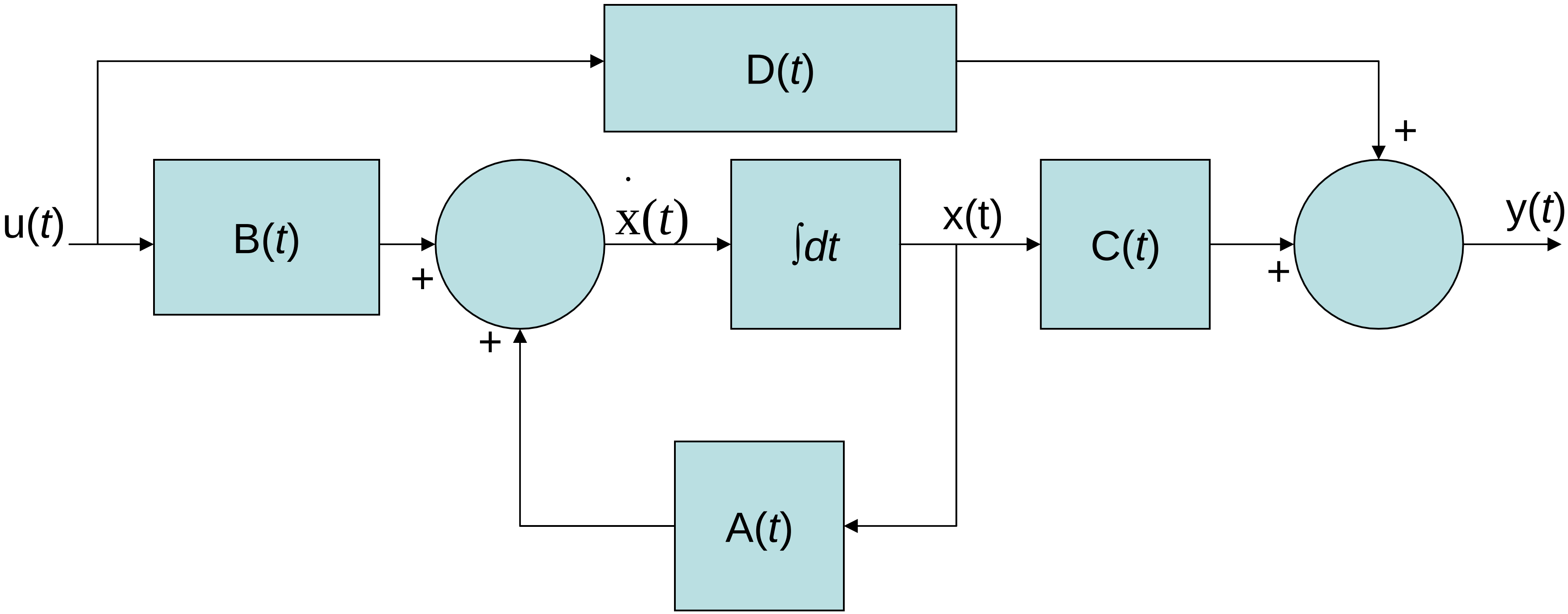
A : State matrix; B : input matrix

C : Output matrix; D : direct transmission matrix

$$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu} \text{ (State differential equation)}$$

$$\mathrm{y} = \mathrm{Cx} + \mathrm{Du} \text{ (Output equation - output signals)}$$

Block Diagram of the Linear, Continuous Time Control System



$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

Mass Grounded, *M* (kg)

Mechanical system described by the first-order differential equation

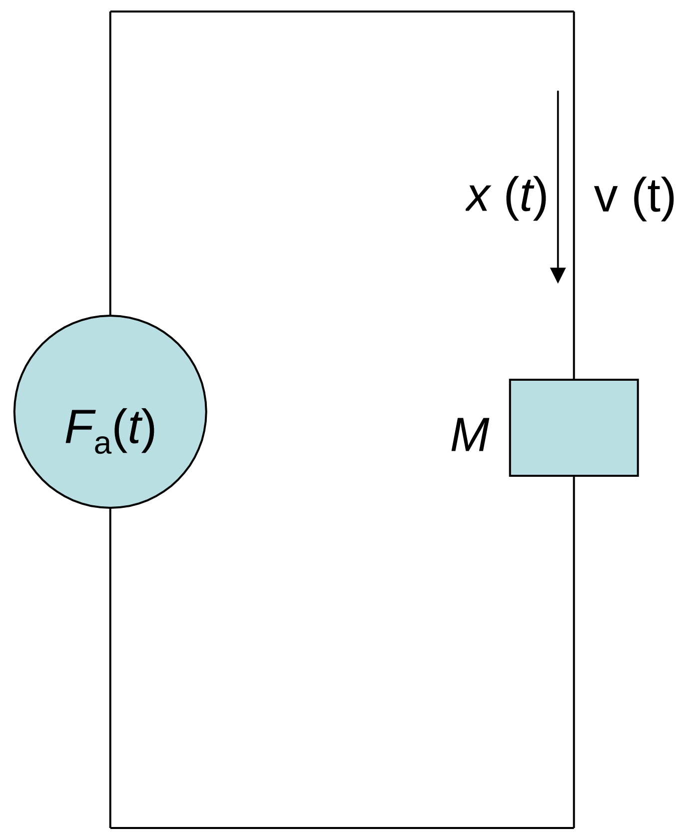
Appied torque $T_{\mathrm{a}}(t)$ (N - m)

Linear velocity $v(t)$ (m/sec)

Linear position $x(t)$ (m)

$$F_a(t) = M\frac{dv}{dt} = M\frac{d^2x(t)}{dt^2}$$

$$v(t) = \frac{1}{M}\int_{t_0}^{t} F_a(t)dt$$



Mechanical Example: Mass-Spring Damper

A set of state variables sufficient to describe this system includes the position and the velocity of the mass, therefore, we will define a set of state variables as (x_1, x_2)

$$x_1(t) = y(t)$$

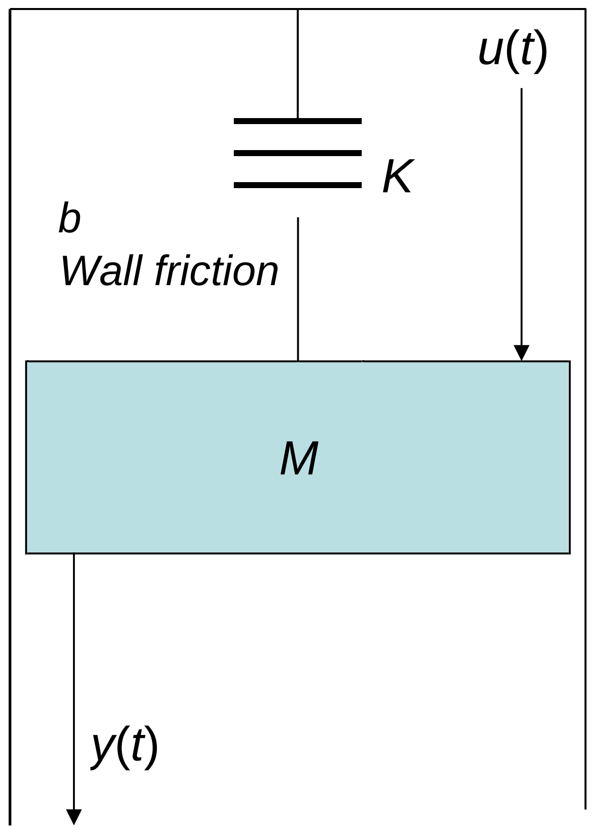
$$x_2(t) = \frac{dy(t)}{dt}$$

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$$

$$\frac{dx_1}{dt} = x_2;$$

$$\frac{dx_2}{dt} = -\frac{b}{m}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$



k :Spring constant

Example 1: Consider the previous mechanical system. Assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output.

The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input-single-output system.

$$m\ddot{y} + b\dot{y} + ky = u$$

This is a second order system. It means it involves two integrators.

Let us define two variables : $x_1(t)$ and $x_2(t)$

$$x_1(t) = y(t);\ x_2(t) = \dot{y}(t);\ \text{then } \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is : $y = x_1$

In a vector matrix form, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \text{ (State Equation)}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ (Output Equation)}$$

The state equation and the output equation are in the standard form :

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}u;\ \mathrm{y} = \mathrm{C}\mathrm{x} + D\mathrm{u}$$

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix},\ \mathrm{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$$

Electrical and Mechanical Counterparts

Energy	Mechanical	Electrical
Kinetic	**Mass / Inertia** $0.5\ mv^2$ / $0.5\ j\omega^2$	**Inductor** $0.5\ Li^2$
Potential	**Gravity:** *mgh* **Spring:** $0.5\ kx^2$	**Capacitor** $0.5\ Cv^2$
Dissipative	**Damper / Friction** $0.5\ Bv^2$	Resistor Ri^2

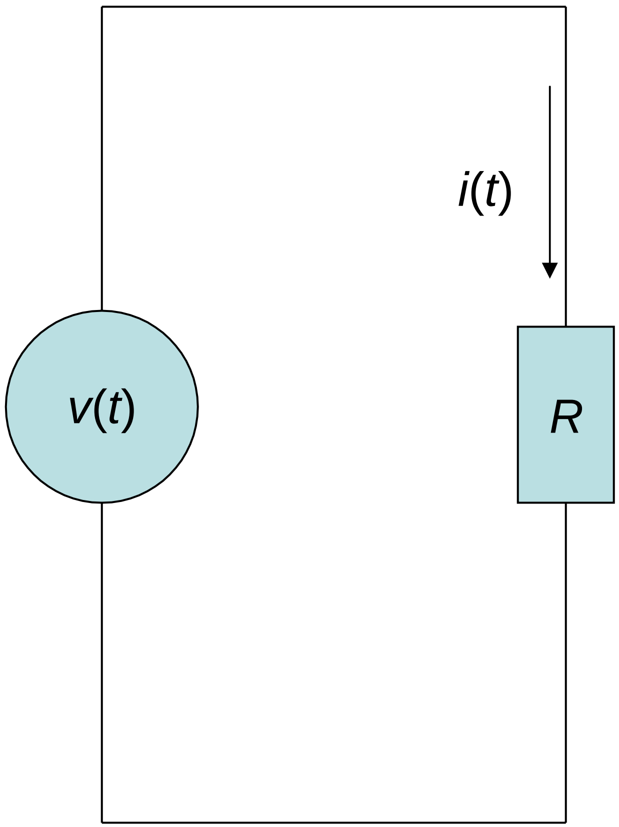
Resistance, *R* (ohm)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = Ri(t)$$

$$i(t) = \frac{1}{R} v(t)$$



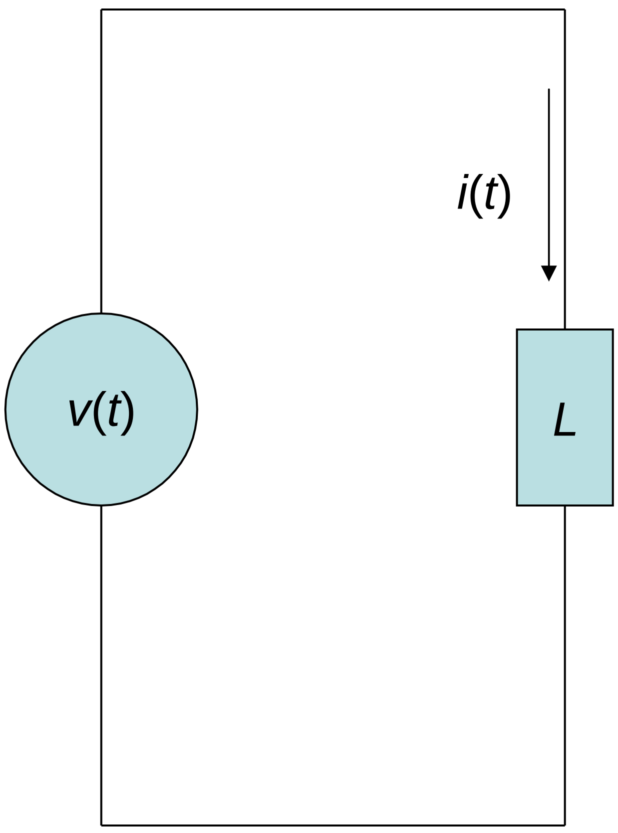
Inductance, *L* (H)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)dt$$



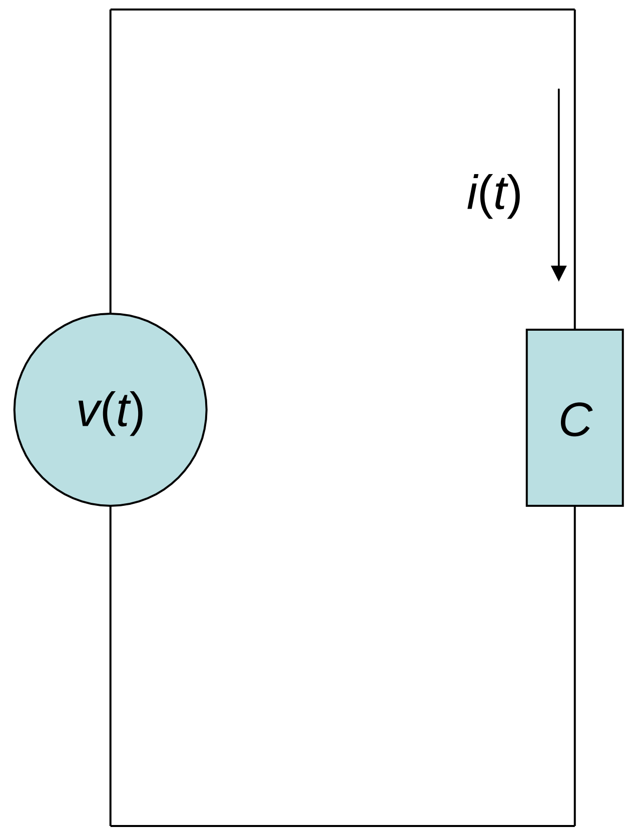
Capacitance, C (F)

Appied voltage $v(t)$

Current $i(t)$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$i(t) = C\,\frac{dv(t)}{dt}$$



Electrical Example: An *RLC* Circuit

$x_1 = v_C(t);\ x_2 = i_L(t)$

$\xi = (1/2)Li_L^2 + (1/2)Cv_c^2$

$x_1(t_0)$ and $x_2(t_0)$ is the total initial energy of the network

USE KCL at the junction

$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

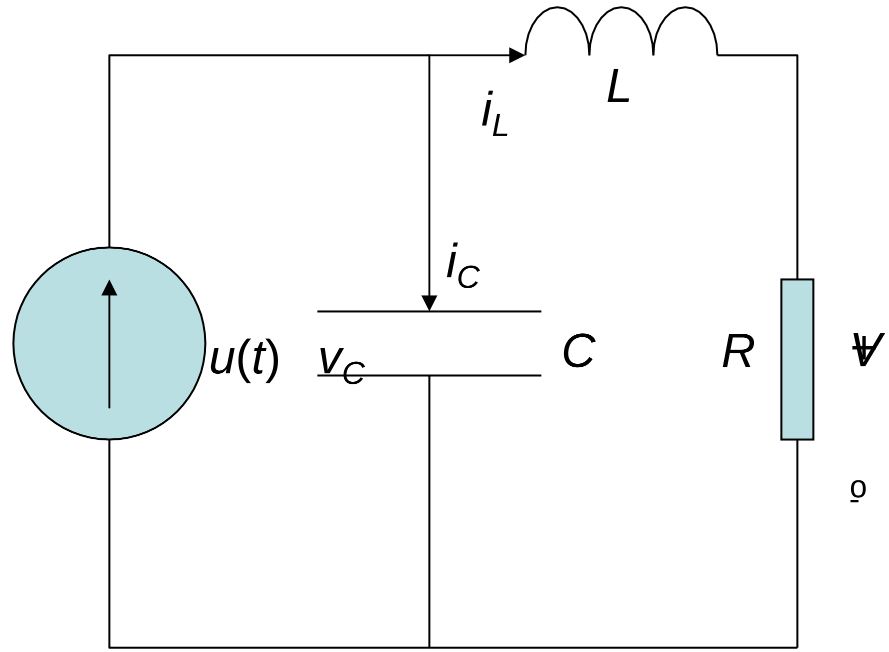
$$L\frac{di_L}{dt} = -Ri_L + v_c$$

The output of the system is represented by : $v_o = Ri_L(t)$

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$\frac{dx_2}{dt} + \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then : $y_1(t) = v_o(t) = Rx_2$



Example 2: Use Equations from the *RLC* circuit

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

The output is

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

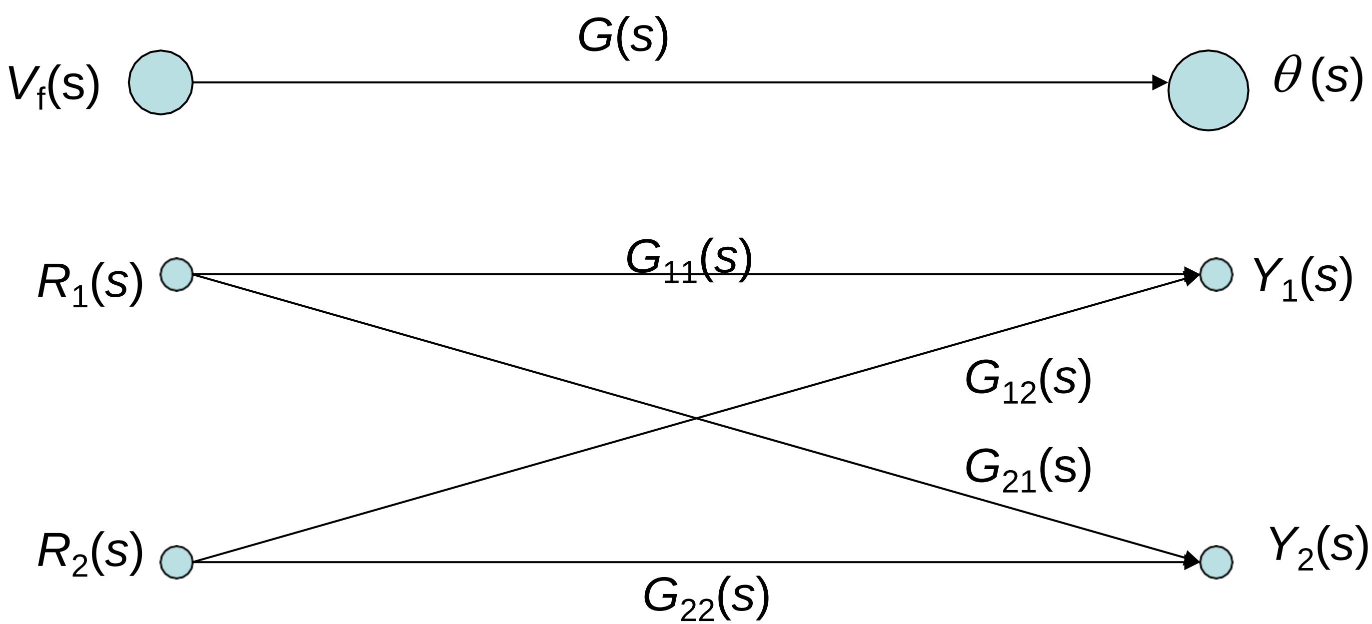
When $R = 3,\ L = 1, C = 1/2$, we have

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} x$$

Signal-Flow Graph Model

A signal-flow graph is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations. Signal-flow graphs are important for feedback systems because feedback theory is concerned with the flow and processing of signals in system.



Read Examples : 2.8 - 2.11

Mason's Gain Formula for Signal Flow Graphs

In many applications, we wish to determine the relationship between an input and output variable of the signal flow diagram. The transmittance between an input node and output node is the overall gain between these two nodes.

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

P_k = path gain of k $_{\text{th}}$ forward path

Δ = determinant of graph

= 1- (sum of all individual loop gain) +

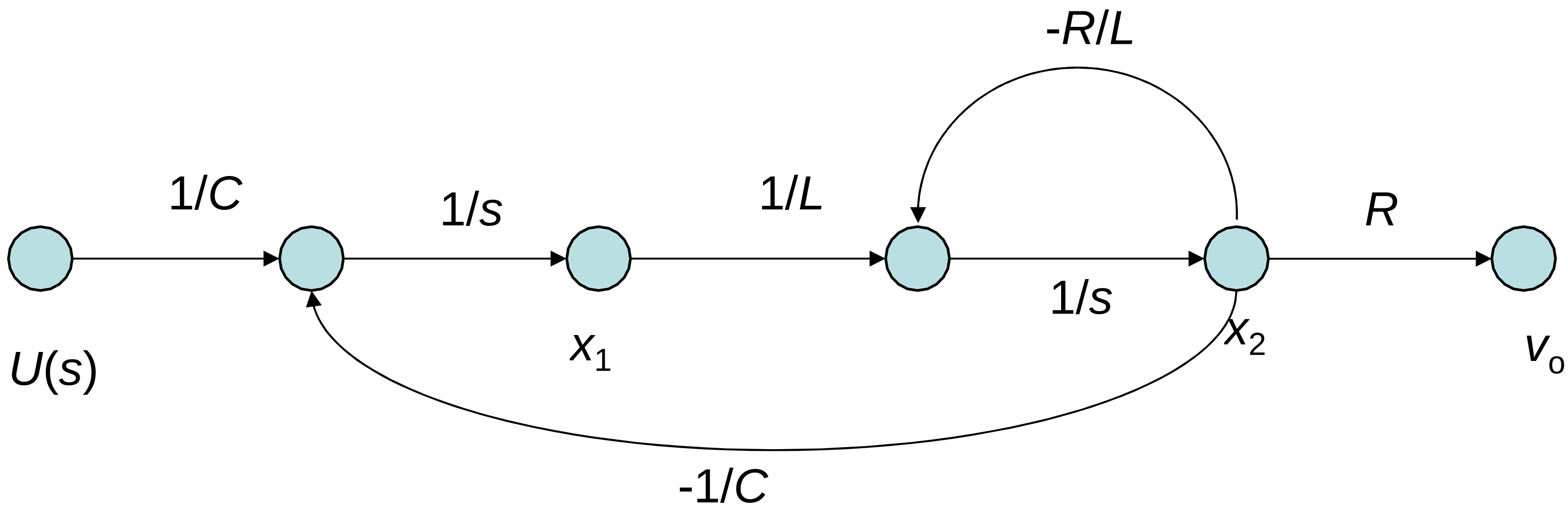
(sum of gain of all possible combinations of two nontouching loops)

- (sumof gain products of all possible combinations of these nontouching loops) + ..

$$= 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_c L_f$$

Δ_k = cofactorof the kth forward path determinant of the graph with theloops touching the kth forward path removed, that is, the cofactor Δ_k is obtained fromΔ by removing the loops that touch path P_k .

Signal-Flow Graph State Models



$$G(s) = \frac{V_o(s)}{U(s)} = \frac{\square\alpha}{s^2 + \beta s + \gamma}$$

$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2 ; v_o = Rx_2$$

$$\frac{Vo(s)}{U(s)} = \frac{R/LCs^2}{1+\left(R/Ls\right)+\left(1/LCs^2\right)} ;= \frac{R/LC}{s^2 + \left(R/L\right)s + \left(1/LC\right)}$$

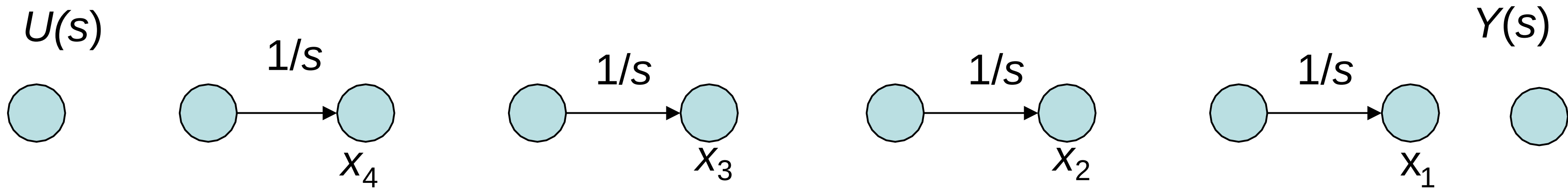
$$G(s)=\frac{Y(s)}{U(s)}=\frac{s^m+b_{m-1}s^{m-1}+\dots+b_1s+b_o}{s^n+a_{n-1}s^{n-1}+\dots+a_1s+a_o}$$

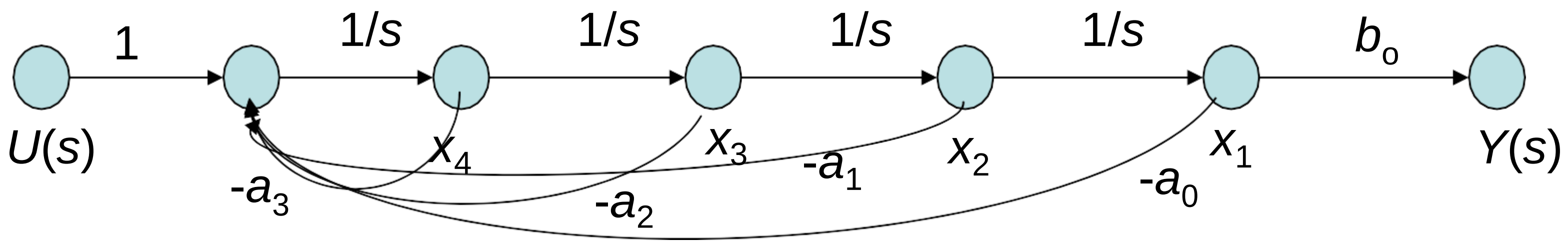
$$G(s)=\frac{Y(s)}{U(s)}=\frac{s^{-(n-m)}+b_{m-1}s^{-(n-m+1)}+\dots+b_1s^{-(n-1)}+b_os^{-n}}{1+a_{n-1}s^{-1}+\dots+a_1s^{-(n-1)}+a_os^{-n}}$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\sum_k P_k\Delta k}{\Delta}$$

$$G(s)=\frac{\sum_k P_k}{1-\sum_{q=1}^{N}Lq}=\frac{\text{Some of the forward - path factors}}{\text{1- sumof the feedback loop factor}}$$

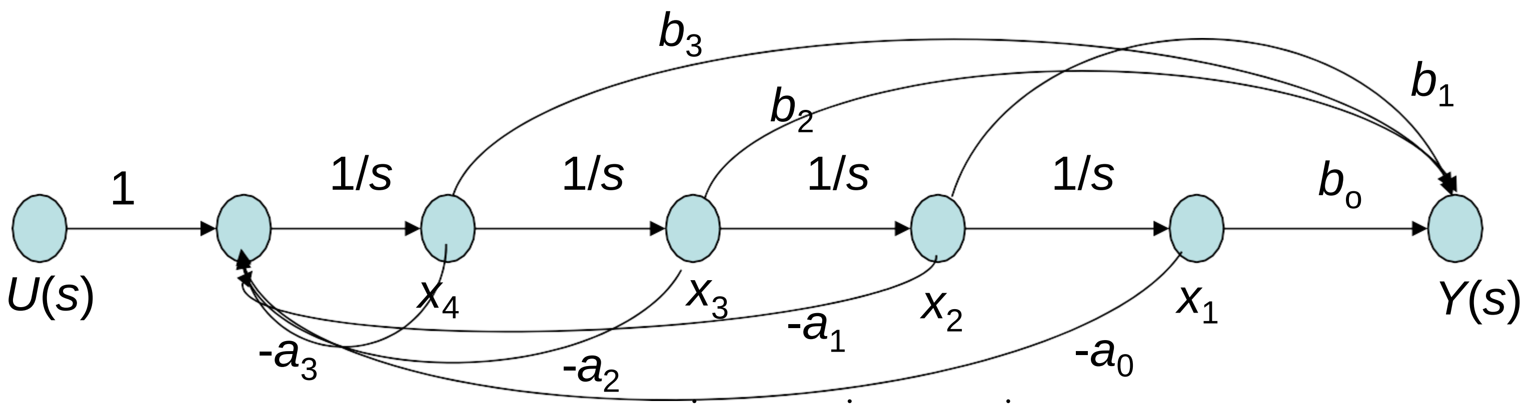
Phase Variable Format: Let us initially consider the fourth-order transfer function. Four state variables (x_1, x_2, x_3, x_4); Number of integrators equal the order of the system.





$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$



$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} = \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

$$\dot{x}_1 = x_2 \; ; \; \dot{x}_2 = x_3 \; ; \; \dot{x}_3 = x_4$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$

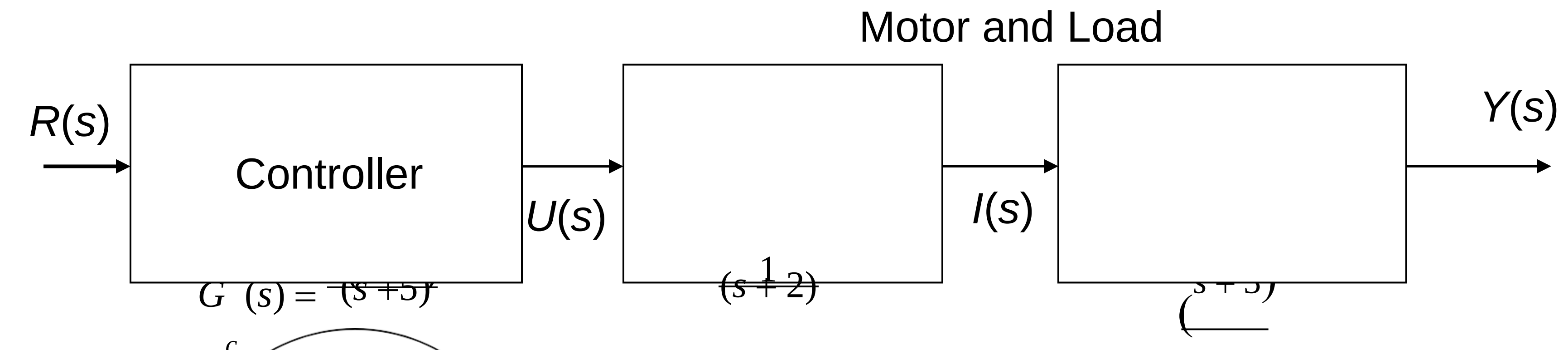
$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

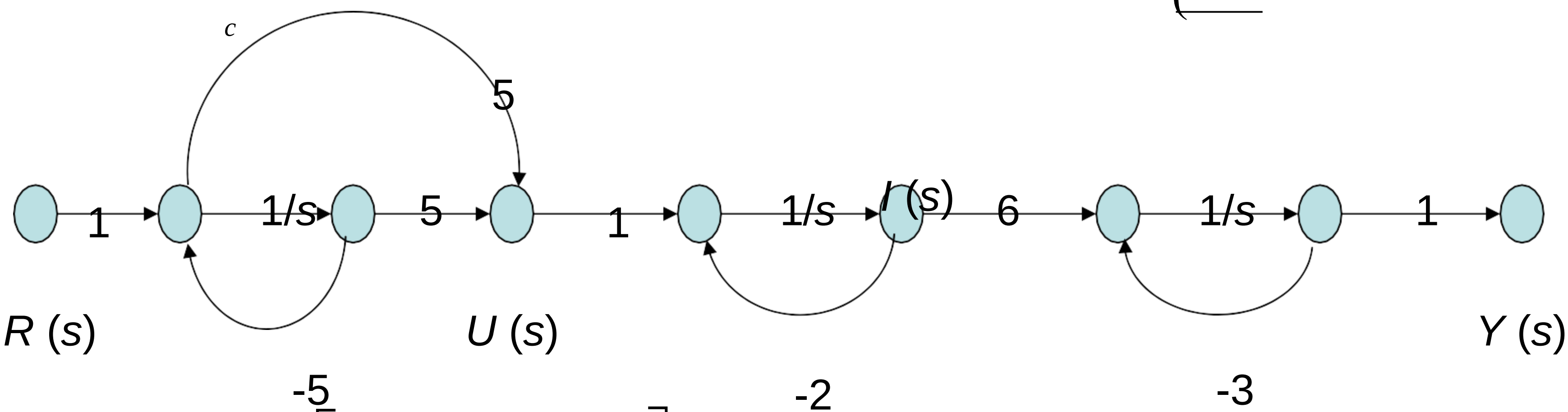
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Read Examplev 3.1of the textbook

$$y(t) = \mathrm{Cx} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

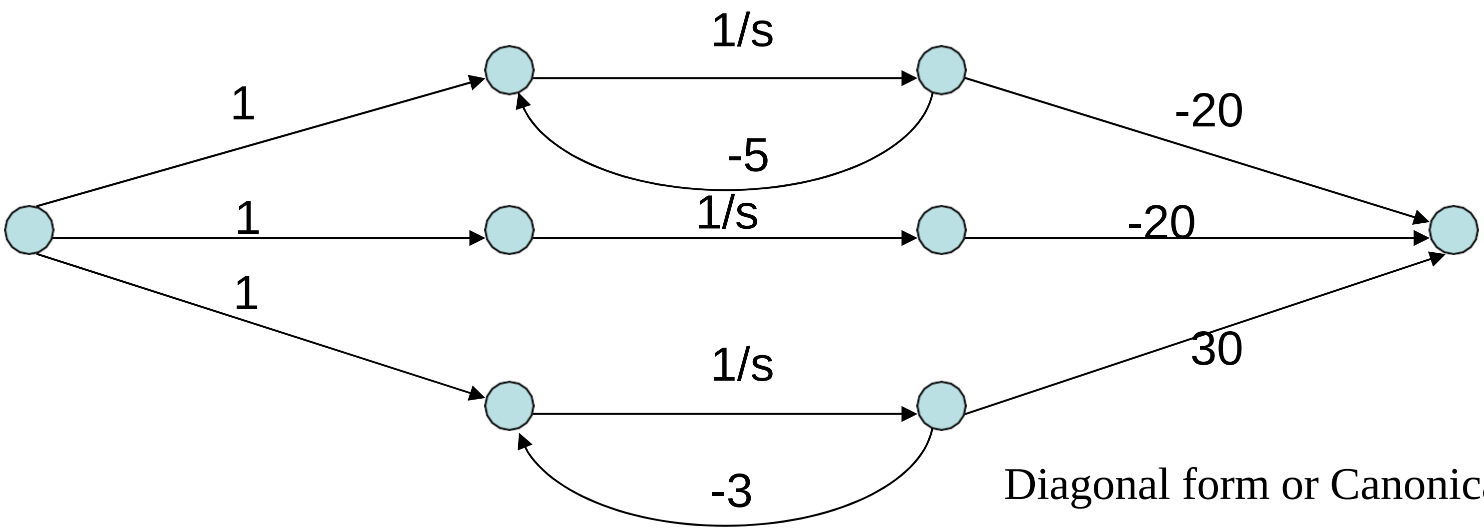
Alternative Signal-Flow Graph State Models





$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t) \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

The State Variable Differential Equations



Diagonal form or Canonical form

$$\frac{Y(s)}{R(s)} = T(s) = \frac{30(s+1)}{(s+5)(s+2)(s+3)} = \frac{q(s)}{(s-s_1)(s-s_2)(s-s_3)}$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{(s+5)} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)}$$

$k_1 = -20$, $k_2 = -10$, and $k_3 = 30$

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t); \quad y(t) = \begin{bmatrix} -20 & -10 & 30 \end{bmatrix} x$$

The State Variable Differential Equations

$$\dot{x} = \begin{bmatrix} -3 & 6 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} r(t)$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{30(s+1)}{(s+5)(s+2)(s+3)} = \frac{q(s)}{(s-s_1)(s-s_2)(s-s_3)}$$

$$\frac{Y(s)}{R(s)} = T(s) = \frac{k_1}{(s+5)} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)}$$

$k_1 = -20$, $k_2 = -10$, and $k_3 = 30$

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} -20 & -10 & 30 \end{bmatrix} x$$

The Transfer Function from the State Equation

Given the transfer function $G(s)$, we may obtain the state variable equations using the signal-flow graph model. Recall the two basic equations

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}u$$

$$y = \mathrm{C}\mathrm{x}$$

y is the single output and u is the single input.

Take the Laplace transform

$$s\mathrm{X}(s) = \mathrm{A}\,\mathrm{X}(s) + \mathrm{B}U(s)$$

$$Y(s) = \mathrm{C}\mathrm{X}(s)$$

$$\left(s\mathrm{I} - \mathrm{A}\right)\mathrm{X}(s) = \mathrm{B}U(s)$$

$$\text{Since } \left[s\mathrm{I} - \mathrm{A}\right]^{-1} = \Phi(s)$$

$$\mathrm{X}(s) = \Phi(s)\,\mathrm{B}U(s)$$

$$Y(s) = \mathrm{C}\,\Phi(s)\,\mathrm{B}U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{C}\,\Phi(s)\mathrm{B}$$

Exercises: E3.2 (DGD)

A robot-arm drive system for one joint can be represented by the differential equation,

$$\frac{dv(t)}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

where $v(t)$ = velocity, $y(t)$ = position, and $i(t)$ is the control-motor current. Put the equations in state variable form and set up the matrix form for $k_1=k_2=1$

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = -k_1 v(t) - k_2 y(t) + k_3 i(t)$$

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i$$

Define $u = i$, and let $k_1 = k_2 = 1$

$$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu};\ \mathrm{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 0 \\ k_3 \end{bmatrix},\ \mathrm{x} = \begin{pmatrix} y \\ v \end{pmatrix}$$

E3.3: A system can be represented by the state vector differential equation of equation (3.16) of the textbook. Find the **characteristic roots** of the system (DGD).

$$\dot{x} = Ax + Bu \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\text{Det}\,(\lambda I - A) = \text{Det}\left(\begin{bmatrix} \lambda & -1 \\ 1 & (\lambda+1) \end{bmatrix}\right)$$

$$= \lambda(\lambda+1)+1 = \lambda^2 + \lambda + 1 = 0$$

$$\angle_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2};\ \lambda_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

E3.7: Consider the spring and mass shown in Figure 3.3 where *M* = 1 kg, *k* = 100 N/m, and *b* = 20 N/m/sec. (a) Find the state vector differential equation. (b) Find the roots of the characteristic equation for this system (DGD).

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -100x_1 - 20x_2 + u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\mathrm{Det}\,(\lambda \mathrm{I} - \mathrm{A}) = \mathrm{Det} \begin{bmatrix} \lambda & -1 \\ 100 & \lambda + 20 \end{bmatrix} = \lambda^2 + 20\lambda + 100$$

$$= (\lambda + 10)^2 = 0;\ \lambda_1 = \lambda_2 = -10$$

E3.8: The manual, low-altitude hovering task above a moving land deck of a small ship is very demanding, in particular, in adverse weather and sea conditions. The hovering condition is represented by the **A** matrix (DGD)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\mathrm{Det}(\lambda I - A) = \mathrm{Det}\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 5 & \lambda = 2 \end{bmatrix} =$$

$$\lambda\left(\lambda^2 + 2\lambda + 5)\right) = 0$$

$$\lambda_1 = 0;\ \lambda_2 = -1 + j2;\ \lambda_3 = -1 - j2$$

E3.9: See the textbook (DGD)

$$\dot{x}_1 = \frac{1}{2}x_2 - x_1$$

$$\dot{x}_2 = -x_1 - x_2$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1/2 \\ 1 & -3/2 \end{bmatrix}\mathbf{x},\ \mathbf{y} = \begin{bmatrix} 1 & -3/2 \end{bmatrix}\mathbf{x}$$

$$s^2 + \frac{5}{2}s + 1 = (s+2)\left(s + \frac{1}{2}\right) = 0$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix}\mathbf{z};\ \mathbf{y} = \begin{bmatrix} -0.35 & -1.79 \end{bmatrix}\mathbf{z}$$

P3.1 (DGD-ELG4152):

Apply KVL

$$v(t) = Ri(t) + L\frac{di}{dt} + +v_c$$

$$v_c = \frac{1}{C}\int idt$$

(a) Select the state variables as $x_1 = i$ and $x_2 = v_c$

(b) The state equations are :

$$\dot{x}_1 = \frac{1}{L}v - \frac{R}{L}x_1 - \frac{1}{L}x_2$$

$$\dot{x}_2 = \frac{1}{C}x_1$$

$$(c)\ \dot{\mathrm{x}} = \begin{bmatrix} -\mathrm{R}/L & -1/L \\ 1/C & 0 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}u$$

FOURIER SERIES AND INTEGRALS

FOURIER SERIES FOR PERIODIC FUNCTIONS

This section explains three Fourier series: sines, cosines, and exponentials e^{ikx}. Square waves (1 or 0 or -1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

Start with $\sin x$. It has period 2π since $\sin(x + 2\pi) = \sin x$. It is an odd function since $\sin(-x) = -\sin x$, and it vanishes at $x = 0$ and $x = \pi$. Every function $\sin nx$ has those three properties, and Fourier looked at *infinite combinations of the sines*:

Fourier sine series $\quad S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$

If the numbers $b_1, b_2, \ldots$ drop off quickly enough (we are foreshadowing the importance of the decay rate) then the sum $S(x)$ will inherit all three properties:

Periodic $\quad S(x + 2\pi) = S(x) \qquad$ Odd $\quad S(-x) = -S(x) \qquad S(0) = S(\pi) = 0$

200 years ago, Fourier startled the mathematicians in France by suggesting that *any function* $S(x)$ with those properties could be expressed as an infinite series of sines. This idea started an enormous development of Fourier series. Our first step is to compute from $S(x)$ the number b_k that multiplies $\sin kx$.

Suppose $S(x) = \sum b_n \sin nx$. *Multiply both sides by* $\sin kx$. *Integrate from* 0 *to* π:

$$\int_0^{\pi} S(x) \sin kx\, dx = \int_0^{\pi} b_1 \sin x \sin kx\, dx + \cdots + \int_0^{\pi} b_k \sin kx \sin kx\, dx + \cdots \quad (2)$$

On the right side, all integrals are zero except the highlighted one with $n = k$. This property of "orthogonality" will dominate the whole chapter. The sines make 90° angles in function space, when their inner products are integrals from 0 to π:

Orthogonality $\quad \int_0^{\pi} \sin nx \sin kx\, dx = 0 \quad \text{if} \quad n \neq k. \quad (3)$

Zero comes quickly if we integrate $\int_0^{\pi} \cos mx\, dx = \left[\frac{\sin mx}{m}\right]_0^{\pi} = 0 - 0$. So we use this:

$$\text{Product of sines} \qquad \sin nx \sin kx = \frac{1}{2}\cos(n-k)x - \frac{1}{2}\cos(n+k)x. \tag{4}$$

Integrating $\cos mx$ with $m = n-k$ and $m = n+k$ proves orthogonality of the sines.

The exception is when $n = k$. Then we are integrating $(\sin kx)^2 = \frac{1}{2} - \frac{1}{2}\cos 2kx$:

$$\int_0^{\pi} \sin kx \sin kx dx = \int_0^{\pi} \frac{1}{2} dx - \int_0^{\pi} \frac{1}{2} \cos 2kx dx = \frac{\pi}{2}. \tag{5}$$

The highlighted term in equation (2) is $b_k\pi/2$. Multiply both sides of (2) by $2/\pi$:

Sine coefficients
$S(-x) = -S(x)$

$$b_k = \frac{2}{\pi}\int_0^{\pi} S(x) \sin kx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x) \sin kx dx. \tag{6}$$

Notice that $S(x) \sin kx$ is *even* (equal integrals from $-\pi$ to 0 and from 0 to π).

I will go immediately to the most important example of a Fourier sine series. $S(x)$ is an odd square wave with $SW(x) = 1$ for $0 < x < \pi$. It is drawn in Figure 4.1 as an odd function (with period 2π) that vanishes at $x = 0$ and $x = \pi$.

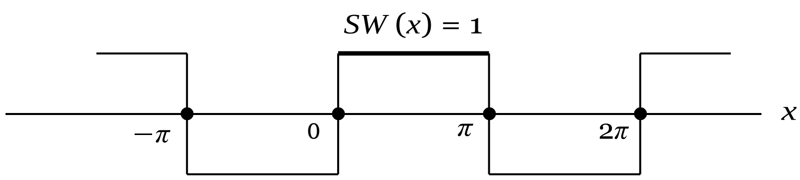


Figure 4.1: The odd square wave with $SW(x + 2\pi) = SW(x) = \{1 \text{ or } 0 \text{ or } -1\}$.

Example 1 Find the Fourier sine coefficients b_k of the square wave $SW(x)$.

Solution For $k = 1, 2, \ldots$ use the first formula (6) with $S(x) = 1$ between 0 and π:

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin kx dx = \frac{2}{\pi}\left[\frac{-\cos kx}{k}\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{2}{1}, \frac{0}{2}, \frac{2}{3}, \frac{0}{4}, \frac{2}{5}, \frac{0}{6}, \ldots\right\} \tag{7}$$

The even-numbered coefficients b_{2k} are all zero because $\cos 2k\pi = \cos 0 = 1$. The odd-numbered coefficients $b_k = 4/\pi k$ decrease at the rate $1/k$. We will see that same $1/k$ decay rate for all functions formed from *smooth pieces and jumps*.

Put those coefficients $4/\pi k$ and zero into the Fourier sine series for $SW(x)$:

$$\text{Square wave} \qquad SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right] \tag{8}$$

Figure 4.2 graphs this sum after one term, then two terms, and then five terms. You can see the all-important Gibbs phenomenon appearing as these “partial sums” include more terms. Away from the jumps, we safely approach $SW(x) = 1$ or -1. At $x = \pi/2$, the series gives a beautiful alternating formula for the number π:

$$1 = \frac{4}{\pi}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \quad \text{so that} \quad \pi = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \tag{9}$$

The Gibbs phenomenon is the overshoot that moves closer and closer to the jumps. Its height approaches $1.18\ldots$ and it does not decrease with more terms of the series! Overshoot is the one greatest obstacle to calculation of all discontinuous functions (like shock waves in fluid flow). We try hard to avoid Gibbs but sometimes we can't.

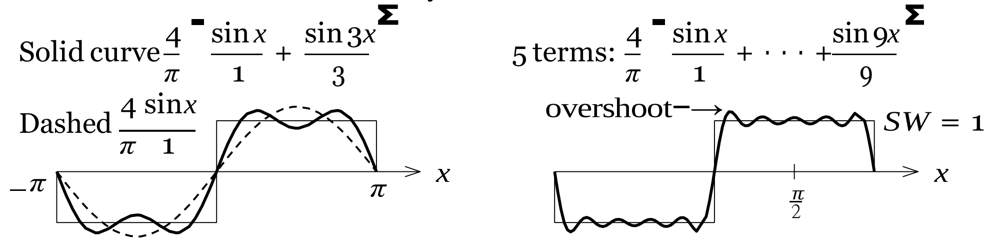


Figure 4.2: Gibbs phenomenon: Partial sums $\sum_1^N b_n \sin nx$ overshoot near jumps.

Fourier Coefficients are Best

Let me look again at the first term $b_1 \sin x = (4/\pi)\sin x$. This is the closest possible approximation to the square wave SW, by any multiple of $\sin x$ (closest in the least squares sense). To see this optimal property of the Fourier coefficients, minimize the error over all b_1:

The error is $\displaystyle\int_0^\pi (SW - b_1 \sin x)^2\,dx$ The b_1 derivative is $\displaystyle -2\int_0^\pi (SW - b_1 \sin x)\sin x\,dx$.

The integral of $\sin^2 x$ is $\pi/2$. So the derivative is zero when $b_1 = (2/\pi)\displaystyle\int_0^\pi S(x)\sin x\,dx$. This is exactly equation (6) for the Fourier coefficient.

Each $b_k \sin kx$ is as close as possible to $SW(x)$. We can find the coefficients b_k one at a time, *because the sines are orthogonal*. The square wave has $b_2 = 0$ because all other multiples of $\sin 2x$ increase the error. Term by term, we are "projecting the function onto each axis $\sin kx$."

Fourier Cosine Series

The cosine series applies to *even functions* with $C(-x) = C(x)$:

$$\text{Cosine series} \quad C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots = a_0 + \sum_{n=1}^{\infty} a_n \cos nx. \tag{10}$$

Every cosine has period 2π. Figure 4.3 shows two even functions, the repeating ramp $RR(x)$ and the up-down train $UD(x)$ of delta functions. That sawtooth ramp RR is the integral of the square wave. The delta functions in UD give the derivative of the square wave. (For sines, the integral and derivative are cosines.) RR and UD will be valuable examples, one smoother than SW, one less smooth.

First we find formulas for the cosine coefficients a_0 and a_k. The constant term a_0 is the *average value* of the function $C(x)$:

$$a_0 = \text{Average} \qquad a_0 = \frac{1}{\pi}\int_0^\pi C(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(x)\,dx. \tag{11}$$

I just integrated every term in the cosine series (10) from 0 to π. On the right side, the integral of a_0 is $a_0\pi$ (divide both sides by π). All other integrals are zero:

$$\int_0^\pi \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^\pi = 0 - 0 = 0. \tag{12}$$

In words, the constant function 1 is orthogonal to $\cos nx$ over the interval $[0, \pi]$.

The other cosine coefficients a_k come from the *orthogonality of cosines*. As with sines, we multiply both sides of (10) by $\cos kx$ and integrate from 0 to π:

$$\int_0^\pi C(x)\cos kx\,dx = \int_0^\pi a_0 \cos kx\,dx + \int_0^\pi a_1 \cos x \cos kx\,dx + \cdots + \int_0^\pi a_k(\cos kx)^2\,dx + \cdots$$

You know what is coming. On the right side, only the highlighted term can be nonzero. Problem 4.1.1 proves this by an identity for $\cos nx \cos kx$—now (4) has a plus sign. The bold nonzero term is $a_k\pi/2$ and we multiply both sides by $2/\pi$:

$$\text{Cosine coefficients } C(-x) = C(x) \qquad a_k = \frac{2}{\pi}\int_0^\pi C(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} C(x)\cos kx\,dx. \tag{13}$$

Again the integral over a full period from $-\pi$ to π (also 0 to 2π) is just doubled.

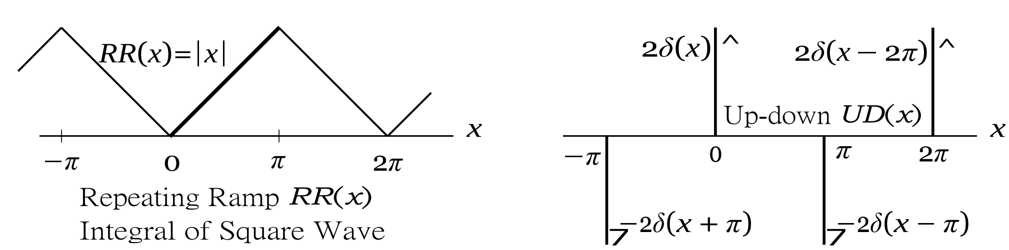


Figure 4.3: The repeating ramp RR and the up-down UD (periodic spikes) are even. The derivative of RR is the odd square wave SW. The derivative of SW is UD.

Example 2 Find the cosine coefficients of the ramp $RR(x)$ and the up-down $UD(x)$.

Solution The simplest way is to start with the sine series for the square wave:

$$SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right].$$

Take the derivative of every term to produce cosines in the up-down delta function:

$$\text{Up-down series} \quad UD(x) = \frac{4}{\pi}\left[\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots\right]. \tag{14}$$

Those coefficients don't decay at all. The terms in the series don't approach zero, so officially the series cannot converge. Nevertheless it is somehow correct and important. Unofficially this sum of cosines has all 1's at $x = 0$ and all -1's at $x = \pi$. Then $+\infty$ and $-\infty$ are consistent with $2\delta(x)$ and $-2\delta(x-\pi)$. The true way to recognize $\delta(x)$ is by the test $\int \delta(x) f(x)\,dx = f(0)$ and Example 3 will do this.

For the repeating ramp, we integrate the square wave series for $SW(x)$ and add the average ramp height $a_0 = \pi/2$, halfway from 0 to π:

$$\text{Ramp series} \quad RR(x) = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right]. \tag{15}$$

The constant of integration is a_0. *Those coefficients a_k drop off like $1/k^2$.* They could be computed directly from formula (13) using $\int x \cos kx\,dx$, but this requires an integration by parts (or a table of integrals or an appeal to *Mathematica* or *Maple*). It was much easier to integrate every sine separately in $SW(x)$, which makes clear the crucial point: Each "degree of smoothness" in the function is reflected in a faster decay rate of its Fourier coefficients a_k and b_k.

No decay	Delta functions (with spikes)
$1/k$ decay	Step functions (with jumps)
$1/k^2$ decay	Ramp functions (with corners)
$1/k^4$ decay	Spline functions (jumps in f''')
r^k decay with $r < 1$	Analytic functions like $1/(2 - \cos x)$

Each integration divides the kth coefficient by k. So the decay rate has an extra $1/k$. The "Riemann-Lebesgue lemma" says that a_k and b_k approach zero for any continuous function (in fact whenever $\int |f(x)|\,dx$ is finite). Analytic functions achieve a new level of smoothness—they can be differentiated forever. Their Fourier series and Taylor series in Chapter 5 converge exponentially fast.

The poles of $1/(2 \cos x)$ will be complex solutions of $\cos x = 2$. Its Fourier series converges quickly because r^k decays faster than any power $1/k^p$. Analytic functions are ideal for computations—the Gibbs phenomenon will never appear.

Now we go back to $\delta(x)$ for what could be the most important example of all.

Example 3 Find the (cosine) coefficients of the *delta function* $\delta(x)$, made 2π-periodic.

Solution The spike occurs at the start of the interval $[0,\pi]$ so safer to integrate from $-\pi$ to π. We find $a_0 = 1/2\pi$ and the other $a_k = 1/\pi$ (cosines because $\delta(x)$ is even):

$$\text{Average } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)\,dx = \frac{1}{2\pi} \qquad \text{Cosines } a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos kx\,dx = \frac{1}{\pi}$$

Then the series for the delta function has all cosines in equal amounts:

$$\text{Delta function} \qquad \delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\cos x + \cos 2x + \cos 3x + \cdots\right]. \tag{16}$$

Again this series cannot truly converge (its terms don't approach zero). But we can graph the sum after $\cos 5x$ and after $\cos 10x$. Figure 4.4 shows how these "partial sums" are doing their best to approach $\delta(x)$. They oscillate faster and faster away from $x = 0$.

Actually there is a neat formula for the partial sum $\delta_N(x)$ that stops at $\cos Nx$. Start by writing each term $2\cos\theta$ as $e^{i\theta} + e^{-i\theta}$:

$$\delta_N = \frac{1}{2\pi}\left[1 + 2\cos x + \cdots + 2\cos Nx\right] = \frac{1}{2\pi}\left[1 + e^{ix} + e^{-ix} + \cdots + e^{iNx} + e^{-iNx}\right].$$

This is a geometric progression that starts from e^{-iNx} and ends at e^{iNx}. We have powers of the same factor e^{ix}. The sum of a geometric series is known:

$$\begin{array}{l}\text{Partial sum}\\ \text{up to } \cos Nx\end{array} \qquad \delta_N(x) = \frac{1}{2\pi}\,\frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \frac{1}{2\pi}\,\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}. \tag{17}$$

This is the function graphed in Figure 4.4. We claim that for any N the area underneath $\delta_N(x)$ is 1. (Each cosine integrated from $-\pi$ to π gives zero. The integral of $1/2\pi$ is 1.) The central "lobe" in the graph ends when $\sin(N+\frac{1}{2})x$ comes down to zero, and that happens when $(N+\frac{1}{2})x = \pm\pi$. I think the area under that lobe (marked by bullets) approaches the same number $1.18\ldots$ that appears in the Gibbs phenomenon.

In what way does $\delta_N(x)$ approach $\delta(x)$? The terms $\cos nx$ in the series jump around at each point $x = 0$, not approaching zero. At $x = \pi$ we see $\frac{1}{2\pi}[1 - 2 + 2 - 2 + \cdots]$ and the sum is $1/2\pi$ or $-1/2\pi$. The bumps in the partial sums don't get smaller than $1/2\pi$. The right test for the delta function $\delta(x)$ is to multiply by a smooth $f(x) = \sum a_k \cos kx$ and integrate, because *we only know* $\delta(x)$ *from its integrals* $\int \delta(x)f(x)\,dx = f(0)$:

$$\begin{array}{l}\text{Weak convergence}\\ \text{of } \delta_N(x) \text{ to } \delta(x)\end{array} \qquad \int_{-\pi}^{\pi} \delta_N(x)f(x)\,dx = a_0 + \cdots + a_N \to f(0)\,. \tag{18}$$

In this integrated sense (*weak sense*) the sums $\delta_N(x)$ do approach the delta function! The convergence of $a_0 + \cdots + a_N$ is the statement that at $x = 0$ the Fourier series of a smooth $f(x) = \sum a_k \cos kx$ converges to the number $f(0)$.

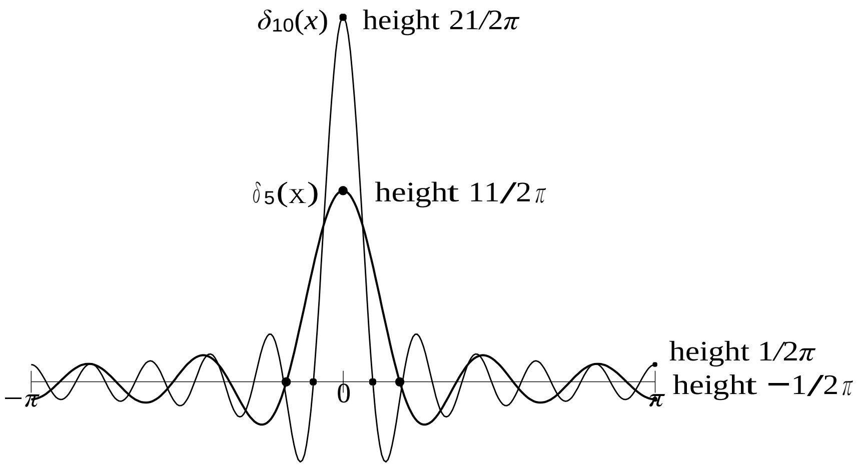


Figure 4.4: The sums $\delta_N(x) = (1+2\cos x + \cdots + 2\cos Nx)/2\pi$ try to approach $\delta(x)$.

Complete Series: Sines and Cosines

Over the half-period $[0, \pi]$, the sines are not orthogonal to all the cosines. In fact the integral of $\sin x$ times 1 is not zero. So for functions $F(x)$ that are not odd or even, we move to the complete series (sines plus cosines) on the full interval. Since our functions are periodic, that "full interval" can be $[-\pi, \pi]$ or $[0, 2\pi]$:

Complete Fourier series

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \qquad (19)$$

On every "2π interval" all sines and cosines are mutually orthogonal. We find the Fourier coefficients a_k and b_k in the usual way: Multiply (19) by 1 and $\cos kx$ and $\sin kx$, and integrate both sides from $-\pi$ to π:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx. \qquad (20)$$

Orthogonality kills off infinitely many integrals and leaves only the one we want.

Another approach is to split $F(x) = C(x) + S(x)$ into an even part and an odd part. Then we can use the earlier cosine and sine formulas. The two parts are

$$C(x) = F_{\text{even}}(x) = \frac{F(x)+F(-x)}{2} \qquad S(x) = F_{\text{odd}}(x) = \frac{F(x)-F(-x)}{2}. \qquad (21)$$

The even part gives the a's and the odd part gives the b's. Test on a short square pulse from $x = 0$ to $x = h$—this one-sided function is not odd or even.

Example 4 Find the a's and b's if $F(x) = \text{square pulse} = \begin{cases} 1 & \text{for } 0 < x < h \\ 0 & \text{for } h < x < 2\pi \end{cases}$

Solution The integrals for a_0 and a_k and b_k stop at $x = h$ where $F(x)$ drops to zero. The coefficients decay like $1/k$ because of the jump at $x = 0$ and the drop at $x = h$:

$$\text{Coefficients of square pulse} \qquad a_0 = \frac{1}{2\pi}\int_0^h 1\,dx = \frac{h}{2\pi} = \text{average}$$

$$a_k = \frac{1}{\pi}\int_0^h \cos kx\,dx = \frac{\sin kh}{\pi k} \qquad b_k = \frac{1}{\pi}\int_0^h \sin kx\,dx = \frac{1 - \cos kh}{\pi k}. \tag{22}$$

If we divide $F(x)$ by h, its graph is a tall thin rectangle: height $\frac{1}{h}$, base h, and area $= 1$.

When h approaches zero, $F(x)/h$ is squeezed into a very thin interval. *The tall rectangle approaches (weakly) the delta function $\delta(x)$.* The average height is area$/2\pi = 1/2\pi$. Its other coefficients a_k/h and b_k/h approach $1/\pi$ and 0, already known for $\delta(x)$:

$$\frac{F(x)}{h} \longrightarrow \delta(x) \qquad \frac{a_k}{h} = \frac{1}{\pi}\frac{\sin kh}{kh} \rightarrow \frac{1}{\pi} \quad \text{and} \quad \frac{b_k}{h} = \frac{1 - \cos kh}{\pi kh} \rightarrow 0 \text{ as } h \rightarrow 0. \tag{23}$$

When the function has a jump, its Fourier series picks the halfway point. This example would converge to $F(0) = \frac{1}{2}$ and $F(h) = \frac{1}{2}$, halfway up and halfway down.

The Fourier series converges to $F(x)$ at each point where the function is smooth. This is a highly developed theory, and Carleson won the 2006 Abel Prize by proving convergence for every x except a set of measure zero. If the function has finite energy $\int |F(x)|^2\,dx$, he showed that the Fourier series converges "almost everywhere."

Energy in Function = Energy in Coefficients

There is an extremely important equation (*the energy identity*) that comes from integrating $(F(x))^2$. When we square the Fourier series of $F(x)$, and integrate from $-\pi$ to π, all the "cross terms" drop out. The only nonzero integrals come from 1^2 and $\cos^2 kx$ and $\sin^2 kx$, multiplied by a_0^2 and a_k^2 and b_k^2:

$$\text{Energy in } F(x) = \int_{-\pi}^{\pi}\left(a_0 + \sum a_k \cos kx + \sum b_k \sin kx\right)^2 dx$$
$$\int_{-\pi}^{\pi} (F(x))^2\,dx = 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots). \tag{24}$$

The energy in $F(x)$ equals the energy in the coefficients. The left side is like the length squared of a vector, except *the vector is a function*. The right side comes from an infinitely long vector of a's and b's. The lengths are equal, which says that the Fourier transform from function to vector is like an orthogonal matrix. Normalized by constants $\sqrt{2\pi}$ and $\sqrt{\pi}$, we have an *orthonormal basis in function space*.

What is this function space ? It is like ordinary 3-dimensional space, except the "vectors" are functions. Their length $\|f\|$ comes from integrating instead of adding: $\|f\|^2 = \int |f(x)|^2 dx$. These functions fill Hilbert space. The rules of geometry hold:

Length $\|f\|^2 = (f, f)$ comes from the inner product $(f, g) = \int f(x)g(x)\,dx$

Orthogonal functions $(f, g) = 0$ produce a right triangle: $\|f + g\|^2 = \|f\|^2 + \|g\|^2$

I have tried to draw Hilbert space in Figure 4.5. It has infinitely many axes. *The energy identity* (24) *is exactly the Pythagoras Law in infinite-dimensional space*.

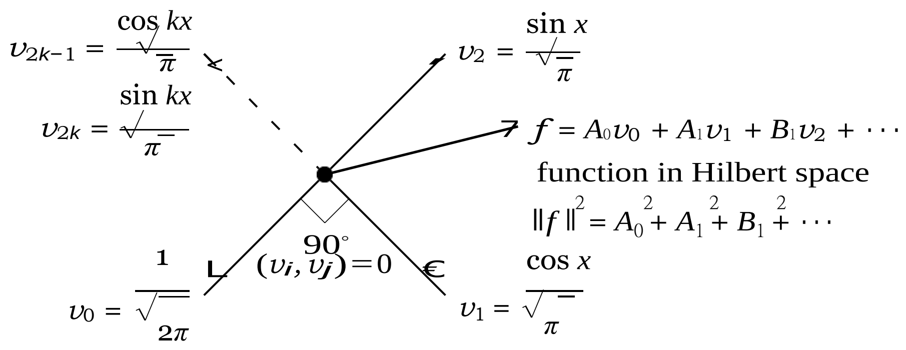


Figure 4.5: The Fourier series is a combination of orthonormal v's (sines and cosines).

Complex Exponentials $c_k e^{ikx}$

This is a small step and we have to take it. In place of separate formulas for a_0 and a_k and b_k, we will have *one formula* for all the complex coefficients c_k. And the function $F(x)$ might be complex (as in quantum mechanics). The Discrete Fourier Transform will be much simpler when we use N complex exponentials for a vector. We practice in advance with the complex infinite series for a 2π-periodic function:

Complex Fourier series $\quad F(x) = c_0 + c_1e^{ix} + c_{-1}e^{-ix} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (25)$

If every $c_n = c_{-n}$, we can combine e^{inx} with e^{-inx} into $2\cos nx$. Then (25) is the cosine series for an even function. If every $c_n = -c_{-n}$, we use $e^{inx} - e^{-inx} = 2i \sin nx$. Then (25) is the sine series for an odd function and the c's are pure imaginary.

To find c_k, multiply (25) by e^{-ikx} (not e^{ikx}) and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} F(x)e^{-ikx}dx = \int_{-\pi}^{\pi} c_0e^{-ikx}dx + \int_{-\pi}^{\pi} c_1e^{ix}e^{-ikx}dx + \cdots + \int_{-\pi}^{\pi} c_k e^{ikx}e^{-ikx}dx + \cdots$$

The complex exponentials are orthogonal. Every integral on the right side is zero, except for the highlighted term (when $n = k$ and $e^{ikx}e^{-ikx} = 1$). The integral of 1 is 2π. That surviving term gives the formula for c_k:

Fourier coefficients $\quad \int_{-\pi}^{\pi} F(x)e^{-ikx}\,dx = 2\pi c_k \quad$ for $k = 0, \pm 1, \ldots \quad (26)$

Notice that $c_0 = a_0$ is still the average of $F(x)$, because $e^0 = 1$. The orthogonality of e^{inx} and e^{ikx} is checked by integrating, as always. But the complex inner product (F, G) takes the *complex conjugate* $\overline{G}$ *of* G. Before integrating, change e^{ikx} to e^{-ikx}:

$$\text{Complex inner product} \quad (F, G) = \int_{-\pi}^{\pi} F(x)\overline{G(x)}\,dx \qquad \text{Orthogonality of } e^{inx} \text{ and } e^{ikx} \quad \int_{-\pi}^{\pi} e^{i(n-k)x}dx = \left[\frac{e^{i(n-k)x}}{i(n-k)}\right]_{-\pi}^{\pi} = 0\,. \tag{27}$$

Example 5 Add the complex series for $1/(2 - e^{ix})$ and $1/(2 - e^{-ix})$. These geometric series have exponentially fast decay from $1/2^k$. The functions are analytic.

$$\left[\frac{1}{2} + \frac{e^{ix}}{4} + \frac{e^{2ix}}{8} + \cdots\right] + \left[\frac{1}{2} + \frac{e^{-ix}}{4} + \frac{e^{-2ix}}{8} + \cdots\right] = 1 + \frac{\cos x}{2} + \frac{\cos 2x}{4} + \frac{\cos 3x}{8} + \cdots$$

When we add those functions, we get a real analytic function:

$$\frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}} = \frac{(2 - e^{-ix}) + (2 - e^{ix})}{(2 - e^{ix})(2 - e^{-ix})} = \frac{4 - 2\cos x}{5 - 4\cos x} \tag{28}$$

This ratio is the infinitely smooth function whose cosine coefficients are $1/2^k$.

Example 6 Find c_k for the 2π-periodic shifted pulse $F(x) = \begin{cases} 1 \text{ for } s \le x \le s + h \\ 0 \text{ elsewhere in } [-\pi, \pi] \end{cases}$

Solution The integrals (26) from $-\pi$ to π become integrals from s to $s + h$:

$$c_k = \frac{1}{2\pi}\int_{s}^{s+h} 1 \cdot e^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{s}^{s+h} = e^{-iks}\left(\frac{1 - e^{-ikh}}{2\pi ik}\right). \tag{29}$$

Notice above all the simple effect of the shift by s. *It "modulates" each* c_k *by* e^{-iks}. The energy is unchanged, the integral of $|F|^2$ just shifts, and all $|e^{-iks}| = 1$:

$$\text{Shift } F(x) \text{ to } F(x - s) \longleftrightarrow \text{Multiply } c_k \text{ by } e^{-iks}. \tag{30}$$

Example 7 Centered pulse with shift $s = -h/2$. The square pulse becomes centered around $x = 0$. This even function equals 1 on the interval from $-h/2$ to $h/2$:

$$\text{Centered by } s = -\frac{h}{2} \qquad c_k = e^{ikh/2}\frac{1 - e^{-ikh}}{2\pi ik} = \frac{1}{2\pi}\frac{\sin(kh/2)}{k/2}\,.$$

Divide by h for a tall pulse. The ratio of $\sin(kh/2)$ to $kh/2$ is the sinc function:

$$\text{Tall pulse} \qquad \frac{F_{\text{centered}}}{h} = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{kh}{2}\right) e^{ikx} = \begin{cases} 1/h \text{ for } -h/2 \le x \le h/2 \\ 0 \quad \text{elsewhere in } [-\pi, \pi] \end{cases}$$

That division by h produces area $= 1$. Every coefficient approaches $\frac{1}{2\pi}$ as $h \to 0$. The Fourier series for the tall thin pulse again approaches the Fourier series for $\delta(x)$.

Hilbert space can contain vectors $c = (c_0, c_1, c_{-1}, c_2, c_{-2}, \cdots)$ instead of functions $F(x)$. The length of c is $2\pi \sum |c_k|^2 = \int |F|^2 dx$. The function space is often denoted by L^2 and the vector space is ℓ^2. The energy identity is trivial (but deep). Integrating the Fourier series for $F(x)$ times $\overline{F(x)}$, orthogonality kills every $c_n \overline{c_k}$ for $n \neq k$. This leaves the $c_k \overline{c_k} = |c_k|^2$:

$$\int_{-\pi}^{\pi} |F(x)|^2 \, dx = \int_{-\pi}^{\pi} \left(\sum c_n e^{inx}\right)\left(\sum \overline{c_k} e^{-ikx}\right) dx = 2\pi(|c_0|^2 + |c_1|^2 + |c_{-1}|^2 + \cdots). \quad (31)$$

This is Plancherel's identity: The energy in x-space equals the energy in k-space.

Finally I want to emphasize the three big rules for operating on $F(x) = \sum c_k e^{ikx}$:

1. The derivative $\dfrac{dF}{dx}$ has Fourier coefficients ikc_k (energy moves to high k).
2. The integral of $F(x)$ has Fourier coefficients $\dfrac{c_k}{ik}, k \neq 0$ (faster decay).
3. The shift to $F(x-s)$ has Fourier coefficients $e^{-iks}c_k$ (no change in energy).

Application: Laplace's Equation in a Circle

Our first application is to Laplace's equation. The idea is to construct $u(x, y)$ as an infinite series, choosing its coefficients to match $u_0(x,y)$ along the boundary. Everything depends on the shape of the boundary, and we take a circle of radius 1.

Begin with the simple solutions $1, r\cos\theta, r\sin\theta, r^2\cos 2\theta, r^2\sin 2\theta, \ldots$ to Laplace's equation. Combinations of these special solutions give all solutions in the circle:

$$\boxed{u(r, \theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \cdots} \quad (32)$$

It remains to choose the constants a_k and b_k to make $u = u_0$ on the boundary. For a circle $u_0(\theta)$ is periodic, since θ and $\theta + 2\pi$ give the same point:

$$\text{Set } r = 1 \qquad u_0(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \cdots \quad (33)$$

This is exactly the Fourier series for u_0. The constants a_k and b_k must be the Fourier coefficients of $u_0(\theta)$. Thus the problem is completely solved, if an infinite series (32) is acceptable as the solution.

Example 8 Point source $u_0 = \delta(\theta)$ at $\theta = 0$ The whole boundary is held at $u_0 = 0$, except for the source at $x = 1, y = 0$. Find the temperature $u(r,\theta)$ inside.

$$\text{Fourier series for } \delta \qquad u_0(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}(\cos\theta + \cos 2\theta + \cos 3\theta + \cdots) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{in\theta}$$

Inside the circle, each $\cos n\theta$ is multiplied by r^n:

$$\text{Infinite series for } u \qquad u(r,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}(r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots) \quad (34)$$

Poisson managed to sum this infinite series! It involves a series of powers of $re^{i\theta}$. So we know the response at every (r,θ) to the point source at $r = 1$, $\theta = 0$:

$$\text{Temperature inside circle} \qquad u(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2r\cos\theta} \quad (35)$$

At the center $r = 0$, this produces the average of $u_0 = \delta(\theta)$ which is $a_0 = 1/2\pi$. On the boundary $r = 1$, this produces $u = 0$ except at the point source where $\cos 0 = 1$:

$$\text{On the ray } \theta = 0 \qquad u(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2r} = \frac{1}{2\pi}\,\frac{1+r}{1-r}. \quad (36)$$

As r approaches 1, the solution becomes infinite as the point source requires.

Example 9 Solve for any boundary values $u_0(\theta)$ by integrating over point sources.

When the point source swings around to angle ϕ, the solution (35) changes from θ to $\theta - \phi$. Integrate this "Green's function" to solve in the circle:

$$\text{Poisson's formula} \qquad u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(\phi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi \quad (37)$$

Ar $r = 0$ the fraction disappears and u is the average $\int u_0(\phi)d\phi/2\pi$. The steady state temperature at the center is the average temperature around the circle.

Poisson's formula illustrates a key idea. Think of any $u_0(\theta)$ as a circle of point sources. The source at angle $\phi = \theta$ produces the solution inside the integral (37). Integrating around the circle adds up the responses to all sources and gives the response to $u_0(\theta)$.

Example 10 $u_0(\theta) = 1$ on the top half of the circle and $u_0 = -1$ on the bottom half.

Solution The boundary values are the square wave $SW(\theta)$. Its sine series is in (8):

$$\text{Square wave for } u_0(\theta) \qquad SW(\theta) = \frac{4}{\pi}\left[\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right] \quad (38)$$

Inside the circle, multiplying by $r, r^2, r^3, \ldots$ gives fast decay of high frequencies:

$$\text{Rapid decay inside} \qquad u(r,\theta) = \frac{4}{\pi}\left[\frac{r\sin\theta}{1} + \frac{r^3\sin 3\theta}{3} + \frac{r^5\sin 5\theta}{5} + \cdots\right] \quad (39)$$

Laplace's equation has smooth solutions, even when $u_0(\theta)$ is not smooth.

■ WORKED EXAMPLE ■

A hot metal bar is moved into a freezer (zero temperature). The sides of the bar are coated so that heat only escapes at the ends. *What is the temperature* $u(x, t)$ along the bar at time t? It will approach $u = 0$ as all the heat leaves the bar.

Solution The heat equation is $u_t = u_{xx}$. At $t = 0$ the whole bar is at a constant temperature, say $u = 1$. The ends of the bar are at zero temperature for all time $t > 0$. This is an initial-boundary value problem:

Heat equation	$u_t = u_{xx}$ with $u(x, 0) = 1$ and $u(0, t) = u(\pi, t) = 0.$	(40)

Those zero boundary conditions suggest a sine series. Its coefficients depend on t:

Series solution of the heat equation
$$u(x, t) = \sum_{1}^{\infty} b_n(t) \sin nx. \qquad (41)$$

The form of the solution shows separation of variables. In a comment below, we look for products $A(x)B(t)$ that solve the heat equation and the boundary conditions. What we reach is exactly $A(x) = \sin nx$ and the series solution (41).

Two steps remain. First, choose each $b_n(t) \sin nx$ to satisfy the heat equation:

Substitute into $u_t = u_{xx}$	$b_n'(t) \sin nx = -n^2 b_n(t) \sin nx$	$b_n(t) = e^{-n^2 t} b_n(0).$

Notice $b_n' = -n^2 b_n$. Now determine each $b_n(0)$ from the initial condition $u(x, 0) = 1$ on $(0, \pi)$. Those numbers are the Fourier sine coefficients of $SW(x)$ in equation (38):

Box function/square wave	$\sum_{1}^{\infty} b_n(0) \sin nx = 1$	$b_n(0) = \dfrac{4}{\pi n}$ for odd n

This completes the series solution of the initial-boundary value problem:

Bar temperature
$$u(x, t) = \sum_{\text{odd } n} \frac{4}{\pi n} e^{-n^2 t} \sin nx. \qquad (42)$$

For large n (high frequencies) the decay of $e^{-n^2 t}$ is very fast. The dominant term $(4/\pi)e^{-t} \sin x$ for large times will come from $n = 1$. This is typical of the heat equation and all diffusion, that the solution (the temperature profile) becomes very smooth as t increases.

Numerical difficulty I regret any bad news in such a beautiful solution. To compute $u(x, t)$, we would probably truncate the series in (42) to N terms. When that finite series is graphed on the website, serious bumps appear in $u_N(x, t)$. You ask if there is a physical reason but there isn't. The solution should have maximum temperature at the midpoint $x = \pi/2$, and decay smoothly to zero at the ends of the bar.

Those unphysical bumps are precisely the Gibbs phenomenon. The initial $u(x,0)$ is 1 on $(0,\pi)$ but its odd reflection is -1 on $(-\pi,0)$. That jump has produced the slow $4/\pi n$ decay of the coefficients, with Gibbs oscillations near $x=0$ and $x=\pi$. The sine series for $u(x,t)$ is not a success numerically. Would finite differences help?

Separation of variables We found $b_n(t)$ as the coefficient of an eigenfunction $\sin nx$. Another good approach is to put $u = A(x)\,B(t)$ directly into $u_t = u_{xx}$:

$$\text{Separation}\quad A(x)\,B'(t) = A''(x)\,B(t) \text{ requires } \frac{A''(x)}{A(x)} = \frac{B'(t)}{B(t)} = \text{constant}. \quad (43)$$

A''/A is constant in space, B'/B is constant in time, and they are equal:

$$\frac{A''}{A} = -\lambda \text{ gives } A = \sin\sqrt{\lambda}\,x \text{ and } \cos\sqrt{\lambda}\,x \qquad \frac{B'}{B} = -\lambda \text{ gives } B = e^{-\lambda t}$$

The products $AB = e^{-\lambda t}\sin\sqrt{\lambda}x$ and $e^{-\lambda t}\cos\sqrt{\lambda}x$ solve the heat equation for any number λ. But the boundary condition $u(0,t)=0$ eliminates the cosines. Then $u(\pi,t)=0$ requires $\lambda = n^2 = 1,4,9,\ldots$ to have $\sin\sqrt{\lambda}\pi = 0$. Separation of variables has recovered the functions in the series solution (42).

Finally $u(x,0)=1$ determines the numbers $4/\pi n$ for odd n. We find zero for even n because $\sin nx$ has $n/2$ positive loops and $n/2$ negative loops. For odd n, the extra positive loop is a fraction $1/n$ of all loops, giving slow decay of the coefficients.

Heat bath (the opposite problem) The solution on the website is $1-u(x,t)$, because it solves a different problem. The bar is initially frozen at $U(x,0) = 0$. It is placed into a heat bath at the fixed temperature $U=1$ (or $U=T_0$). The new unknown is U and its boundary conditions are no longer zero.

The heat equation and its boundary conditions are solved first by $U_B(x,t)$. In this example $U_B \equiv 1$ is constant. Then the difference $V = U - U_B$ has zero boundary values, and its initial values are $V=-1$. Now the eigenfunction method (or separation of variables) solves for V. (The series in (42) is multiplied by -1 to account for $V(x,0)=-1$.) Adding back U_B solves the heat bath problem: $U = U_B + V = 1 - u(x,t)$.

Here $U_B \equiv 1$ is the *steady state* solution at $t=\infty$, and V is the *transient* solution. The transient starts at $V=-1$ and decays quickly to $V=0$.

Heat bath at one end The website problem is different in another way too. The Dirichlet condition $u(\pi,t)=1$ is replaced by the Neumann condition $u'(1,t)=0$. Only the left end is in the heat bath. Heat flows down the metal bar and out at the far end, now located at $x=1$. How does the solution change for fixed-free?

Again $U_B = 1$ is a steady state. The boundary conditions apply to $V = 1 - U_B$:

$$\text{Fixed-free eigenfunctions}\quad V(0)=0 \text{ and } V'(1)=0 \text{ lead to } A(x) = \sin\left(n+\frac{1}{2}\right)\pi x. \quad (44)$$

Those eigenfunctions give a new form for the sum of $B_n(t)\,A_n(x)$:

$$\text{Fixed-free solution} \qquad V(x,t) = \sum_{\text{odd } n} B_n(0)\, e^{-(n+\frac{1}{2})^2\pi^2 t} \sin\left(n+\frac{1}{2}\right)\pi x. \qquad (45)$$

All frequencies shift by $\frac{1}{2}$ and multiply by π, because $A'' = -\lambda A$ has a free end at $x = 1$. The crucial question is: Does orthogonality still hold for these new eigenfunctions $\sin\left(n+\frac{1}{2}\right)\pi x$ on $[0,1]$? The answer is *yes* because this fixed-free "Sturm–Liouville problem" $A'' = -\lambda A$ is still symmetric.

Summary The series solutions all succeed but the truncated series all fail. We can see the overall behavior of $u(x, t)$ and $V(x, t)$. But their exact values close to the jumps are not computed well until we improve on Gibbs.

We could have solved the fixed-free problem on $[0,1]$ with the fixed-fixed solution on $[0, 2]$. That solution will be symmetric around $x = 1$ so its slope there is zero. Then rescaling x by 2π changes $\sin(n+\frac{1}{2})\pi x$ into $\sin(2n+1)x$. I hope you like the graphics created by Aslan Kasimov on the cse website.

Problem Set 4.1

1 Find the Fourier series on $-\pi \le x \le \pi$ for

(a) $f(x) = \sin^3 x$, an odd function

(b) $f(x) = |\sin x|$, an even function

(c) $f(x) = x$

(d) $f(x) = e^x$, using the complex form of the series.

What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

2 From Parseval's formula the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |f(x)|^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Derive the remarkable sum $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

3 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \text{ for } |x| < \frac{\pi}{2}, \quad f(x) = 0 \text{ for } \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

4 Suppose f has period T instead of 2π, so that $f(x) = f(x+T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

$$f(x) = a_0 + a_1 \cos\frac{2\pi x}{T} + b_1 \sin\frac{2\pi x}{T} + \cdots = \sum_{-\infty}^{\infty} c_k e^{ik2\pi x/T}$$

Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find a_k, b_k, and c_k.

5 Plot the first three partial sums and the function itself:

$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), \quad 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is the second derivative?

6 What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

7 Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

8 (a) Find the lengths of the vectors $u = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $v = (1, \frac{1}{3}, \frac{1}{9}, \ldots)$ in Hilbert space and test the Schwarz inequality $|u^{\mathrm{T}}v|^2 \le (u^{\mathrm{T}}u)(v^{\mathrm{T}}v)$.

(b) For the functions $f = 1 + \frac{1}{2}e^{ix} + \frac{1}{4}e^{2ix} + \cdots$ and $g = 1 + \frac{1}{3}e^{ix} + \frac{1}{9}e^{2ix} + \cdots$ use part (a) to find the numerical value of each term in

$$\left|\int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx\right|^2 \le \int_{-\pi}^{\pi} |f(x)|^2\, dx \int_{-\pi}^{\pi} |g(x)|^2\, dx.$$

Substitute for f and g and use orthogonality (or Parseval).

9 Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3 \cdots) = 2\log(1 + z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2\log(1 + z)$ agrees with θ.

10 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin?

11 With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum the series.

12 (a) Verify that the fraction in Poisson's formula satisfies Laplace's equation.

(b) What is the response $u(r, \theta)$ to an impulse at the point $(0, 1)$, at the angle $\phi = \pi/2$?

(c) If $u_0(\phi) = 1$ in the quarter-circle $0 < \phi < \pi/2$ and $u_0 = 0$ elsewhere, show that at points on the horizontal axis (and especially at the origin)

$$u(r, 0) = \frac{1}{2} + \frac{1}{2\pi}\tan^{-1}\left(\frac{1 - r^2}{-2r}\right) \quad \text{by using}$$

$$\int \frac{d\phi}{b + c\cos\phi} = \frac{1}{\sqrt{b^2 - c^2}}\tan^{-1}\left(\frac{\sqrt{b^2 - c^2}\,\sin\phi}{c + b\cos\phi}\right).$$

13 When the centered square pulse in Example 7 has width $h = \pi$, find

(a) its energy $\int |F(x)|^2\, dx$ by direct integration

(b) its Fourier coefficients c_k as specific numbers

(c) the sum in the energy identity (31) or (24)

If $h = 2\pi$, why is $c_0 = 1$ the only nonzero coefficient ? What is $F(x)$?

14 In Example 5, $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is infinitely smooth:

(a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

(b) Does that series still converge quickly? Compare n^{10} with 2^n for n^{1024}.

15 (*A touch of complex analysis*) The analytic function in Example 5 blows up when $4\cos x = 5$. This cannot happen for real x, but equation (28) shows blowup if $e^{ix} = 2$ or $\frac{1}{2}$. In that case we have *poles at* $x = \pm i \log 2$. Why are there also poles at all the complex numbers $x = \pm i \log 2 + 2\pi n$?

16 (*A second touch*) Change 2's to 3's so that equation (28) has $1/(3 - e^{ix}) + 1/(3 - e^{-ix})$. Complete that equation to find the function that gives fast decay at the rate $1/3^k$.

17 (*For complex professors only*) Change those 2's and 3's to 1's:

$$\frac{1}{1-e^{ix}} + \frac{1}{1-e^{-ix}} = \frac{(1-e^{-ix}) + (1-e^{ix})}{(1-e^{ix})(1-e^{-ix})} = \frac{2-e^{ix}-e^{-ix}}{2-e^{ix}-e^{-ix}} = 1.$$

A constant! What happened to the pole at $e^{ix} = 1$? Where is the dangerous series $(1 + e^{ix} + \cdots) + (1 + e^{-ix} + \cdots) = 2 + 2\cos x + \cdots$ involving $\delta(x)$?

18 Following the Worked Example, solve the heat equation $u_t = u_{xx}$ from a point source $u(x, 0) = \delta(x)$ with free boundary conditions $u'(\pi, t) = u'(-\pi, t) = 0$. Use the infinite cosine series for $\delta(x)$ with time decay factors $b_n(t)$.

Laplace Transform

The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the Laplace method is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

The direct Laplace transform or the Laplace integral of a function $f(t)$ defined for $0 \leq t < \infty$ is the ordinary calculus integration problem

$$\int_0^\infty f(t)e^{-st}dt,$$

succinctly denoted $\mathbf{L}(f(t))$ in science and engineering literature. The $\mathbf{L}$–notation recognizes that integration always proceeds over $t = 0$ to $t = \infty$ and that the integral involves an *integrator* $e^{-st}dt$ instead of the usual dt. These minor differences distinguish Laplace integrals from the ordinary integrals found on the inside covers of calculus texts.

Introduction to the Laplace Method

The foundation of Laplace theory is Lerch's cancellation law

$$(1)\qquad \begin{array}{lll} \int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt & \text{implies} & y(t) = f(t), \\ & \text{or} & \\ \mathbf{L}(y(t)) = \mathbf{L}(f(t)) & \text{implies} & y(t) = f(t). \end{array}$$

In differential equation applications, $y(t)$ is the sought-after unknown while $f(t)$ is an explicit expression taken from integral tables.

Below, we illustrate Laplace's method by solving the initial value problem

$$y' = -1, \quad y(0) = 0.$$

The method obtains a relation $\mathbf{L}(y(t)) = \mathbf{L}(-t)$, whence Lerch's cancellation law implies the solution is $y(t) = -t$.

The Laplace method is advertised as a *table lookup method*, in which the solution $y(t)$ to a differential equation is found by looking up the answer in a special integral table.

Laplace Integral. The integral $\int_0^\infty g(t)e^{-st}\,dt$ is called the Laplace integral of the function $g(t)$. It is defined by $\lim_{N\to\infty}\int_0^N g(t)e^{-st}\,dt$ and depends on variable s. The ideas will be illustrated for $g(t)=1$, $g(t)=t$ and $g(t)=t^2$, producing the integral formulas in Table 1.

$\int_0^\infty (1)e^{-st}\,dt = -(1/s)e^{-st}\Big\vert_{t=0}^{t=\infty}$	Laplace integral of $g(t)=1$.
$= 1/s$	Assumed $s>0$.
$\int_0^\infty (t)e^{-st}\,dt = \int_0^\infty -\frac{d}{ds}(e^{-st})\,dt$	Laplace integral of $g(t)=t$.
$= -\frac{d}{ds}\int_0^\infty (1)e^{-st}\,dt$	Use $\int \frac{d}{ds}F(t,s)\,dt = \frac{d}{ds}\int F(t,s)\,dt$.
$= -\frac{d}{ds}(1/s)$	Use $\mathbf{L}(1)=1/s$.
$= 1/s^2$	Differentiate.
$\int_0^\infty (t^2)e^{-st}\,dt = \int_0^\infty -\frac{d}{ds}(te^{-st})\,dt$	Laplace integral of $g(t)=t^2$.
$= -\frac{d}{ds}\int_0^\infty (t)e^{-st}\,dt$	
$= -\frac{d}{ds}(1/s^2)$	Use $\mathbf{L}(t)=1/s^2$.
$= 2/s^3$	

Table 1. The Laplace integral $\int_0^\infty g(t)e^{-st}\,dt$ for $g(t)=1$, t and t^2.

$$\int_0^\infty (1)e^{-st}\,dt = \frac{1}{s} \qquad \int_0^\infty (t)e^{-st}\,dt = \frac{1}{s^2} \qquad \int_0^\infty (t^2)e^{-st}\,dt = \frac{2}{s^3}$$

In summary, $\mathbf{L}(t^n) = \dfrac{n!}{s^{1+n}}$

An Illustration. The ideas of the Laplace method will be illustrated for the solution $y(t)=-t$ of the problem $y'=-1$, $y(0)=0$. The method, entirely different from variation of parameters or undetermined coefficients, uses basic calculus and college algebra; see Table 2.

Table 2. Laplace method details for the illustration $y'=-1$, $y(0)=0$.

$y'(t)e^{-st} = -e^{-st}$	Multiply $y'=-1$ by e^{-st}.
$\int_0^\infty y'(t)e^{-st}\,dt = \int_0^\infty -e^{-st}\,dt$	Integrate $t=0$ to $t=\infty$.
$\int_0^\infty y'(t)e^{-st}\,dt = -1/s$	Use Table 1.
$s\int_0^\infty y(t)e^{-st}\,dt - y(0) = -1/s$	Integrate by parts on the left.
$\int_0^\infty y(t)e^{-st}\,dt = -1/s^2$	Use $y(0)=0$ and divide.
$\int_0^\infty y(t)e^{-st}\,dt = \int_0^\infty (-t)e^{-st}\,dt$	Use Table 1.
$y(t) = -t$	Apply Lerch's cancellation law.

In Lerch's law, the formal rule of erasing the integral signs is valid *provided* the integrals are equal for large s and certain conditions hold on y and f – see Theorem 2. The illustration in Table 2 shows that Laplace theory requires an in-depth study of a special integral table, a table which is a true extension of the usual table found on the inside covers of calculus books. Some entries for the special integral table appear in Table 1 and also in section 7.2, Table 4.

The **L**-notation for the direct Laplace transform produces briefer details, as witnessed by the translation of Table 2 into Table 3 below. The reader is advised to move from Laplace integral notation to the **L**–notation as soon as possible, in order to clarify the ideas of the transform method.

Table 3. Laplace method **L**-notation details for $y' = -1$, $y(0) = 0$ translated from Table 2.

$\mathbf{L}(y'(t)) = \mathbf{L}(-1)$	Apply **L** across $y' = -1$, or multiply $y' = -1$ by e^{-st}, integrate $t = 0$ to $t = \infty$.
$\mathbf{L}(y'(t)) = -1/s$	Use Table 1.
$s\mathbf{L}(y(t)) - y(0) = -1/s$	Integrate by parts on the left.
$\mathbf{L}(y(t)) = -1/s^2$	Use $y(0) = 0$ and divide.
$\mathbf{L}(y(t)) = \mathbf{L}(-t)$	Apply Table 1.
$y(t) = -t$	Invoke Lerch's cancellation law.

Some Transform Rules. The formal properties of calculus integrals plus the integration by parts formula used in Tables 2 and 3 leads to these rules for the Laplace transform:

$\mathbf{L}(f(t) + g(t)) = \mathbf{L}(f(t)) + \mathbf{L}(g(t))$	The integral of a sum is the sum of the integrals.
$\mathbf{L}(cf(t)) = c\mathbf{L}(f(t))$	Constants c pass through the integral sign.
$\mathbf{L}(y'(t)) = s\mathbf{L}(y(t)) - y(0)$	The t-derivative rule, or integration by parts. See Theorem 3.
$\mathbf{L}(y(t)) = \mathbf{L}(f(t))$ implies $y(t) = f(t)$	Lerch's cancellation law. See Theorem 2.

1 **Example (Laplace method)** Solve by Laplace's method the initial value problem $y' = 5 - 2t$, $y(0) = 1$.

Solution: Laplace's method is outlined in Tables 2 and 3. The **L**-notation of Table 3 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$\mathcal{L}(y'(t)) = \mathcal{L}(5-2t)$	Apply $\mathcal{L}$ across $y' = 5-2t$.
$\mathcal{L}(y'(t)) = \frac{5}{s} - \frac{2}{s^2}$	Use Table 1.
$s\mathcal{L}(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2}$	Apply the t-derivative rule, page 248.
$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}$	Use $y(0) = 1$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(1) + 5\mathcal{L}(t) - \mathcal{L}(t^2)$	Apply Table 1, backwards.
$= \mathcal{L}(1 + 5t - t^2)$	Linearity, page 248.
$y(t) = 1 + 5t - t^2$	Invoke Lerch's cancellation law.

2 Example (Laplace method) Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$.

Solution: The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 5t^2$.

$\mathcal{L}(y''(t)) = \mathcal{L}(10)$	Apply $\mathcal{L}$ across $y'' = 10$.
$s\mathcal{L}(y'(t)) - y'(0) = \mathcal{L}(10)$	Apply the t-derivative rule to y', that is, replace y by y' on page 248.
$s[s\mathcal{L}(y(t)) - y(0)] - y'(0) = \mathcal{L}(10)$	Repeat the t-derivative rule, on y.
$s^2\mathcal{L}(y(t)) = \mathcal{L}(10)$	Use $y(0) = y'(0) = 0$.
$\mathcal{L}(y(t)) = \frac{10}{s^3}$	Use Table 1. Then divide.
$\mathcal{L}(y(t)) = \mathcal{L}(5t^2)$	Apply Table 1, backwards.
$y(t) = 5t^2$	Invoke Lerch's cancellation law.

Existence of the Transform. The Laplace integral $\int_0^\infty e^{-st} f(t)\,dt$ is known to exist in the sense of the improper integral definition[1]

$$\int_0^\infty g(t)\,dt = \lim_{N\to\infty} \int_0^N g(t)\,dt$$

provided $f(t)$ belongs to a class of functions known in the literature as functions of *exponential order*. For this class of functions the relation

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \tag{2}$$

is required to hold for some real number a, or equivalently, for some constants M and a,

$$|f(t)| \le Me^{at}. \tag{3}$$

In addition, $f(t)$ is required to be *piecewise continuous* on each finite subinterval of $0 \le t < \infty$, a term defined as follows.

[1] An advanced calculus background is assumed for the Laplace transform existence proof. Applications of Laplace theory require only a calculus background.

Deftnition 1 (piecewise continuous)

A function $f(t)$ is piecewise continuous on a finite interval $[a, b]$ provided there exists a partition $a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$ and functions $f_1, f_2, \ldots, f_n$ continuous on $(-\infty, \infty)$ such that for t not a partition point

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1, \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t < t_n. \end{cases} \tag{4}$$

The values of f at partition points are undecided by equation (4). In particular, equation (4) implies that $f(t)$ has one-sided limits at each point of $a < t < b$ and appropriate one-sided limits at the endpoints. Therefore, f has at worst a jump discontinuity at each partition point.

3 Example (Exponential order) Show that $f(t) = e^t \cos t + t$ is of exponential order, that is, show that $f(t)$ is piecewise continuous and find $a > 0$ such that $\lim_{t\to\infty} f(t)/e^{at} = 0$.

Solution: Already, $f(t)$ is continuous, hence piecewise continuous. From L'Hospital's rule in calculus, $\lim_{t\to\infty} p(t)/e^{at} = 0$ for any polynomial p and any $a > 0$. Choose $a = 2$, then

$$\lim_{t\to\infty} \frac{f(t)}{e^{2t}} = \lim_{t\to\infty} \frac{\cos t}{e^t} + \lim_{t\to\infty} \frac{t}{e^{2t}} = 0.$$

Theorem 1 (Existence of $\mathbf{L}(f)$)

Let $f(t)$ be piecewise continuous on every finite interval in $t \geq 0$ and satisfy $|f(t)| \leq Me^{at}$ for some constants M and a. Then $\mathbf{L}(f(t))$ exists for $s > a$ and $\lim_{s\to\infty} \mathbf{L}(f(t)) = 0$.

Proof: It has to be shown that the Laplace integral of f is finite for $s > a$. Advanced calculus implies that it is sufficient to show that the integrand is absolutely bounded above by an integrable function $g(t)$. Take $g(t) = Me^{-(s-a)t}$. Then $g(t) \geq 0$. Furthermore, g is integrable, because

$$\int_0^\infty g(t)dt = \frac{M}{s-a}.$$

Inequality $|f(t)| \leq Me^{at}$ implies the absolute value of the Laplace transform integrand $f(t)e^{-st}$ is estimated by

$$\left|f(t)e^{-st}\right| \leq Me^{at}e^{-st} = g(t).$$

The limit statement follows from $|\mathbf{L}(f(t))| \leq \int_0^\infty g(t)dt = \dfrac{M}{s-a}$, because the right side of this inequality has limit zero at $s = \infty$. The proof is complete.

Theorem 2 (Lerch)
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and $\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt$ for all $s > s_0$, then $f_1(t) = f_2(t)$ for $t \geq 0$.

Proof: See Widder [?].

Theorem 3 (t-Derivative Rule)
If $f(t)$ is continuous, $\lim_{t\to\infty} f(t)e^{-st} = 0$ for all large values of s and $f'(t)$ is piecewise continuous, then $\mathbf{L}(f'(t))$ exists for all large s and $\mathbf{L}(f'(t)) = s\mathbf{L}(f(t)) - f(0)$.

Proof: See page 276.

Exercises 7.1

Laplace method. Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.
2. $y' = 1$, $y(0) = 0$.
3. $y' = -t$, $y(0) = 0$.
4. $y' = t$, $y(0) = 0$.
5. $y' = 1 - t$, $y(0) = 0$.
6. $y' = 1 + t$, $y(0) = 0$.
7. $y' = 3 - 2t$, $y(0) = 0$.
8. $y' = 3 + 2t$, $y(0) = 0$.
9. $y'' = -2$, $y(0) = y'(0) = 0$.
10. $y'' = 1$, $y(0) = y'(0) = 0$.
11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.
12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.
13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.
14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order. Show that $f(t)$ is of exponential order, by finding a constant $\alpha \geq 0$ in each case such that $\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$
16. $f(t) = e^t \sin(t)$
17. $f(t) = \sum_{n=0}^{N} c_n x^n$ for any choice of the constants $c_0, \ldots, c_N$.
18. $f(t) = \sum_{n=1}^{N} c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms. Let $f(t) = te^{t^2}\sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.
20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude. For f piecewise continuous, define the jump at t by

$$J(t) = \lim_{h\to 0+} f(t+h) - \lim_{h\to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \geq 0$, else $f(t) = 0$
22. $f(t) = 1$ for $t \geq 1/2$, else $f(t) = 0$
23. $f(t) = t/|t|$ for $t \neq 0$, $f(0) = 0$
24. $f(t) = \sin t/|\sin t|$ for $t \neq n\pi$, $f(n\pi) = (-1)^n$

Taylor series. The series relation $\mathbf{L}(\sum_{n=0}^\infty c_n t^n) = \sum_{n=0}^\infty c_n \mathbf{L}(t^n)$ often holds, in which case the result $\mathbf{L}(t^n) = n!s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathbf{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^\infty (2t)^n/n!$
26. $f(t) = e^{-t} = \sum_{n=0}^\infty (-t)^n/n!$

Laplace Integral Table

The objective in developing a table of Laplace integrals, e.g., Tables 4 and 5, is to keep the table size small. Table manipulation rules appearing in Table 6, page 257, effectively increase the table size manyfold, making it possible to solve typical differential equations from electrical and mechanical problems. The combination of Laplace tables plus the table manipulation rules is called the Laplace transform calculus.

Table 4 is considered to be a table of minimum size to be memorized. Table 5 adds a number of special-use entries. For instance, the Heaviside entry in Table 5 is memorized, but usually not the others.

Derivations are postponed to page 270. The theory of the gamma function $\Gamma(x)$ appears below on page 255.

Table 4. A minimal Laplace integral table with L-notation

$\int_0^\infty (t^n)e^{-st}\,dt = \dfrac{n!}{s^{1+n}}$	$\mathcal{L}(t^n) = \dfrac{n!}{s^{1+n}}$
$\int_0^\infty (e^{at})e^{-st}\,dt = \dfrac{1}{s-a}$	$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$
$\int_0^\infty (\cos bt)e^{-st}\,dt = \dfrac{s}{s^2+b^2}$	$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$
$\int_0^\infty (\sin bt)e^{-st}\,dt = \dfrac{b}{s^2+b^2}$	$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

Table 5. Laplace integral table extension

$\mathcal{L}(H(t-a)) = \dfrac{e^{-as}}{s}\ (a \ge 0)$	Heaviside unit step, defined by $H(t) = \begin{cases} 1 & \text{for } t \ge 0, \\ 0 & \text{otherwise.} \end{cases}$
$\mathcal{L}(\delta(t-a)) = e^{-as}$	Dirac delta, $\delta(t) = dH(t)$. Special usage rules apply.
$\mathcal{L}(\text{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$	Staircase function, $\text{floor}(x)$ = greatest integer $\le x$.
$\mathcal{L}(\text{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$	Square wave, $\text{sqw}(x) = (-1)^{\text{floor}(x)}$.
$\mathcal{L}(a\,\text{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$	Triangular wave, $\text{trw}(x) = \int_0^x \text{sqw}(r)\,dr$.
$\mathcal{L}(t^a) = \dfrac{\Gamma(1+a)}{s^{1+a}}$	Generalized power function, $\Gamma(1+a) = \int_0^\infty e^{-x}x^a\,dx$.
$\mathcal{L}(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$	Because $\Gamma(1/2) = \sqrt{\pi}$.

4 Example (Laplace transform) Let $f(t) = t(t-1) - \sin 2t + e^{3t}$. Compute $\mathbf{L}(f(t))$ using the basic Laplace table and transform linearity properties.

Solution:

$\mathrm{L}(f(t)) = \mathrm{L}(t^2 - 5t - \sin 2t + e^{3t})$	Expand $t(t-5)$.
$= \mathrm{L}(t^2) - 5\mathrm{L}(t) - \mathrm{L}(\sin 2t) + \mathrm{L}(e^{3t})$	Linearity applied.
$= \dfrac{2}{s^3} - \dfrac{5}{s^2} - \dfrac{2}{s^2+4} + \dfrac{1}{s-3}$	Table lookup.

5 Example (Inverse Laplace transform) Use the basic Laplace table backwards plus transform linearity properties to solve for $f(t)$ in the equation

$$\mathbf{L}(f(t)) = \frac{s}{s^2+16} + \frac{2}{s-3} + \frac{s+1}{s^3}.$$

Solution:

$\mathrm{L}(f(t)) = \dfrac{s}{s^2+16} + 2\dfrac{1}{s-3} + \dfrac{1}{s^2} + \dfrac{1}{2}\dfrac{2}{s^3}$	Convert to table entries.
$= \mathrm{L}(\cos 4t) + 2\mathrm{L}(e^{3t}) + \mathrm{L}(t) + \frac{1}{2}\mathrm{L}(t^2)$	Laplace table (backwards).
$= \mathrm{L}(\cos 4t + 2e^{3t} + t + \frac{1}{2}t^2)$	Linearity applied.
$f(t) = \cos 4t + 2e^{3t} + t + \frac{1}{2}t^2$	Lerch's cancellation law.

6 Example (Heaviside) Find the Laplace transform of $f(t)$ in Figure 1.

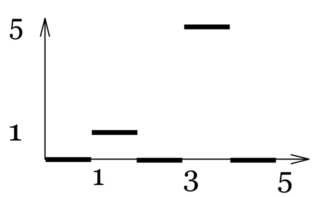


Figure 1. A piecewise deftned function $f(t)$ on $0 \le t < \infty$: $f(t) = 0$ except for $1 \le t < 2$ and $3 \le t < 4$.

Solution: The details require the use of the Heaviside function formula

$$H(t-a) - H(t-b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise}. \end{cases}$$

The formula for $f(t)$:

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 5 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise} \end{cases} + 5 \begin{cases} 1 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases}$$

Then $f(t) = f_1(t) + 5f_2(t)$ where $f_1(t) = H(t-1) - H(t-2)$ and $f_2(t) = H(t-3) - H(t-4)$. The extended table gives

$\mathrm{L}(f(t)) = \mathrm{L}(f_1(t)) + 5\mathrm{L}(f_2(t))$	Linearity.
$= \mathrm{L}(H(t-1)) - \mathrm{L}(H(t-2)) + 5\mathrm{L}(f_2(t))$	Substitute for f_1.

$$= \frac{e^{-s} - e^{-2s}}{s} + 5\mathcal{L}(f_2(t))$$ Extended table used.

$$= \frac{e^{-s} - e^{-2s} + 5e^{-3s} - 5e^{-4s}}{s}$$ Similarly for f_2.

7 Example (Dirac delta) A machine shop tool that repeatedly hammers a die is modeled by the Dirac impulse model $f(t) = \sum_{n=1}^{N} \delta(t - n)$. Show that $\mathcal{L}(f(t)) = \sum_{n=1}^{N} e^{-ns}$.

Solution:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}\left(\sum_{n=1}^{N} \delta(t-n)\right) \\
&= \sum_{n=1}^{N} \mathcal{L}(\delta(t-n)) && \text{Linearity.} \\
&= \sum_{n=1}^{N} e^{-ns} && \text{Extended Laplace table.}
\end{aligned}
$$

8 Example (Square wave) A periodic camshaft force $f(t)$ applied to a mechanical system has the idealized graph shown in Figure 2. Show that $f(t) = 1 + \text{sqw}(t)$ and $\mathcal{L}(f(t)) = \frac{1}{s}(1 + \tanh(s/2))$.

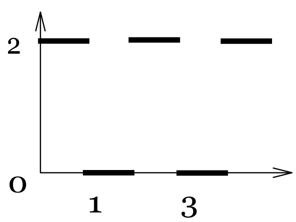


Figure 2. A periodic force $f(t)$ applied to a mechanical system.

Solution:

$$
\begin{aligned}
1 + \text{sqw}(t) &= \begin{cases} 1+1 & 2n \le t < 2n+1,\ n = 0, 1, \ldots, \\ 1-1 & 2n+1 \le t < 2n+2,\ n = 0, 1, \ldots, \end{cases} \\
&= \begin{cases} 2 & 2n \le t < 2n+1,\ n = 0, 1, \ldots, \\ 0 & \text{otherwise,} \end{cases} \\
&= f(t).
\end{aligned}
$$

By the extended Laplace table, $\mathcal{L}(f(t)) = \mathcal{L}(1) + \mathcal{L}(\text{sqw}(t)) = \frac{1}{s} + \frac{\tanh(s/2)}{s}$.

9 Example (Sawtooth wave) Express the P-periodic sawtooth wave represented in Figure 3 as $f(t) = ct/P - c\,\text{floor}(t/P)$ and obtain the formula

$$\mathcal{L}(f(t)) = \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}}.$$

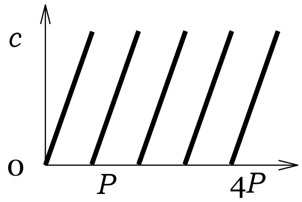


Figure 3. A P-periodic sawtooth wave $f(t)$ of height $c > 0$.

Solution: The representation originates from geometry, because the periodic function f can be viewed as derived from ct/P by subtracting the correct constant from each of intervals $[P, 2P]$, $[2P, 3P]$, etc.

The technique used to verify the identity is to define $g(t) = ct/P - c\,\mathrm{floor}(t/P)$ and then show that g is P-periodic and $f(t) = g(t)$ on $0 \le t < P$. Two P-periodic functions equal on the base interval $0 \le t < P$ have to be identical, hence the representation follows.

The fine details: for $0 \le t < P$, $\mathrm{floor}(t/P) = 0$ and $\mathrm{floor}(t/P + k) = k$. Hence $g(t + kP) = ct/P + ck - c\,\mathrm{floor}(k) = ct/P = g(t)$, which implies that g is P-periodic and $g(t) = f(t)$ for $0 \le t < P$.

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{c}{P}\mathcal{L}(t) - c\mathcal{L}(\mathrm{floor}(t/P)) && \text{Linearity.}\\
&= \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}} && \text{Basic and extended table applied.}
\end{aligned}
$$

10 **Example (Triangular wave)** Express the triangular wave f of Figure 4 in terms of the square wave sqw and obtain $\mathbf{L}(f(t)) = \dfrac{5}{\pi s^2}\tanh(\pi s/2)$.

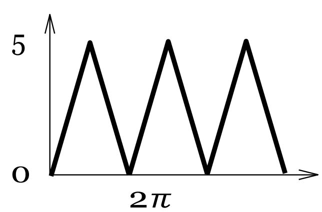


Figure 4. A 2π-periodic triangular wave $f(t)$ of height 5.

Solution: The representation of f in terms of sqw is $f(t) = 5\int_0^{t/\pi} \mathrm{sqw}(x)\,dx$.
Details: A 2-periodic triangular wave of height 1 is obtained by integrating the square wave of period 2. A wave of height c and period 2 is given by $c\,\mathrm{trw}(t) = c\int_0^t \mathrm{sqw}(x)\,dx$. Then $f(t) = c\,\mathrm{trw}(2t/P) = c\int_0^{2t/P} \mathrm{sqw}(x)\,dx$ where $c = 5$ and $P = 2\pi$.
Laplace transform details: Use the extended Laplace table as follows.

$$\mathcal{L}(f(t)) = \frac{5}{\pi}\mathcal{L}(\pi\,\mathrm{trw}(t/\pi)) = \frac{5}{\pi s^2}\tanh(\pi s/2).$$

Gamma Function. In mathematical physics, the Gamma function or the generalized factorial function is given by the identity

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt, \quad x > 0. \tag{1}$$

This function is tabulated and available in computer languages like Fortran, C and C++. It is also available in computer algebra systems and numerical laboratories. Some useful properties of $\Gamma(x)$:

$$\Gamma(1 + x) = x\Gamma(x) \tag{2}$$

$$\Gamma(1 + n) = n! \quad \text{for integers } n \ge 1. \tag{3}$$

Details for relations (2) and (3): Start with $\int_0^\infty e^{-t}\,dt = 1$, which gives $\Gamma(1) = 1$. Use this identity and successively relation (2) to obtain relation (3). To prove identity (2), integration by parts is applied, as follows:

$$\begin{aligned}
\Gamma(1+x) &= \int_0^\infty e^{-t} t^x\,dt && \text{Definition.} \\
&= -t^x e^{-t}\Big|_{t=0}^{t=\infty} + \int_0^\infty e^{-t} x t^{x-1}\,dt && \text{Use } u = t^x,\ dv = e^{-t}dt. \\
&= x\int_0^\infty e^{-t} t^{x-1}\,dt && \text{Boundary terms are zero for } x > 0. \\
&= x\Gamma(x).
\end{aligned}$$

Exercises 7.2

Laplace transform. Compute $\mathcal{L}(f(t))$ using the basic Laplace table and the linearity properties of the transform. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$
2. $\mathcal{L}(4t)$
3. $\mathcal{L}(1+2t+t^2)$
4. $\mathcal{L}(t^2-3t+10)$
5. $\mathcal{L}(\sin 2t)$
6. $\mathcal{L}(\cos 2t)$
7. $\mathcal{L}(e^{2t})$
8. $\mathcal{L}(e^{-2t})$
9. $\mathcal{L}(t+\sin 2t)$
10. $\mathcal{L}(t-\cos 2t)$
11. $\mathcal{L}(t+e^{2t})$
12. $\mathcal{L}(t-3e^{-2t})$
13. $\mathcal{L}((t+1)^2)$
14. $\mathcal{L}((t+2)^2)$
15. $\mathcal{L}(t(t+1))$
16. $\mathcal{L}((t+1)(t+2))$
17. $\mathcal{L}(\sum_{n=0}^{10} t^n/n!)$
18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1}/n!)$
19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$
20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Inverse Laplace transform. Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$
22. $\mathcal{L}(f(t)) = 4s^{-2}$
23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$
24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$
25. $\mathcal{L}(f(t)) = 2/(s^2+4)$
26. $\mathcal{L}(f(t)) = s/(s^2+4)$
27. $\mathcal{L}(f(t)) = 1/(s-3)$
28. $\mathcal{L}(f(t)) = 1/(s+3)$
29. $\mathcal{L}(f(t)) = 1/s + s/(s^2+4)$
30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2+4)$
31. $\mathcal{L}(f(t)) = 1/s + 1/(s-3)$
32. $\mathcal{L}(f(t)) = 1/s - 3/(s-2)$
33. $\mathcal{L}(f(t)) = (2+s)^2/s^3$
34. $\mathcal{L}(f(t)) = (s+1)/s^2$
35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$
36. $\mathcal{L}(f(t)) = (s+1)(s-1)/s^3$
37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$
38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$
39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \dfrac{n}{s^2+n^2}$
40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2+n^2}$

7.3 Laplace Transform Rules

In Table 6, the basic table manipulation rules are summarized. Full statements and proofs of the rules appear in section 7.7, page 275.

The rules are applied here to several key examples. Partial fraction expansions do not appear here, but in section 7.4, in connection with Heaviside's coverup method.

Table 6. Laplace transform rules

$\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	**Linearity.** The Laplace of a sum is the sum of the Laplaces.
$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	**Linearity.** Constants move through the $\mathcal{L}$-symbol.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	**The t-derivative rule.** Derivatives $\mathcal{L}(y')$ are replaced in transformed equations.
$\mathcal{L}\left(\int_0^t g(x)dx\right) = \frac{1}{s}\mathcal{L}(g(t))$	**The t-integral rule.**
$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$	**The s-differentiation rule.** Multiplying f by t applies $-d/ds$ to the transform of f.
$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\big\vert_{s\to(s-a)}$	**First shifting rule.** Multiplying f by e^{at} replaces s by $s-a$.
$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t))$, $\mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a))$	**Second shifting rule.** First and second forms.
$\mathcal{L}(f(t)) = \dfrac{\int_0^P f(t)e^{-st}dt}{1-e^{-Ps}}$	**Rule for P-periodic functions.** Assumed here is $f(t+P)=f(t)$.
$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}((f*g)(t))$	**Convolution rule.** Define $(f*g)(t) = \int_0^t f(x)g(t-x)dx$.

11 **Example (Harmonic oscillator)** Solve by Laplace's method the initial value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$.

Solution: The solution is $x(t) = \sin t$. The details:

$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(0)$	Apply $\mathcal{L}$ across the equation.
$s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = 0$	Use the t-derivative rule.
$s[s\mathcal{L}(x) - x(0)] - x'(0) + \mathcal{L}(x) = 0$	Use again the t-derivative rule.
$(s^2+1)\mathcal{L}(x) = 1$	Use $x(0) = 0$, $x'(0) = 1$.
$\mathcal{L}(x) = \dfrac{1}{s^2+1}$	Divide.
$\quad = \mathcal{L}(\sin t)$	Basic Laplace table.
$x(t) = \sin t$	Invoke Lerch's cancellation law.

12 Example (s-differentiation rule) Show the steps for $\mathbf{L}(t^2 e^{5t}) = \dfrac{2}{(s-5)^3}$.

Solution:

$\mathbf{L}(t^2 e^{5t}) = \left(-\dfrac{d}{ds}\right)\left(-\dfrac{d}{ds}\right)\mathbf{L}(e^{5t})$	Apply s-differentiation.
$= (-1)^2 \dfrac{d}{ds}\dfrac{d}{ds}\left(\dfrac{1}{s-5}\right)$	Basic Laplace table.
$= \dfrac{d}{ds}\left(\dfrac{-1}{(s-5)^2}\right)$	Calculus power rule.
$= \dfrac{2}{(s-5)^3}$	Identity verified.

13 Example (First shifting rule) Show the steps for $\mathbf{L}(t^2 e^{-3t}) = \dfrac{2}{(s+3)^3}$.

Solution:

$\mathbf{L}(t^2 e^{-3t}) = \mathbf{L}(t^2)\Big\vert_{s\to s-(-3)}$	First shifting rule.
$= \left(\dfrac{2}{s^{2+1}}\right)\Big\vert_{s\to s-(-3)}$	Basic Laplace table.
$= \dfrac{2}{(s+3)^3}$	Identity verified.

14 Example (Second shifting rule) Show the steps for

$$\mathbf{L}(\sin t\,H(t-\pi)) = \frac{e^{-\pi s}}{s^2+1}.$$

Solution: The second shifting rule is applied as follows.

$\mathbf{L}(\sin t\,H(t-\pi)) = \mathbf{L}(g(t)H(t-a))$	Choose $g(t) = \sin t$, $a = \pi$.
$= e^{-as}\mathbf{L}(g(t+a))$	Second form, second shifting theorem.
$= e^{-\pi s}\mathbf{L}(\sin(t+\pi))$	Substitute $a = \pi$.
$= e^{-\pi s}\mathbf{L}(-\sin t)$	Sum rule $\sin(a+b) = \sin a \cos b + \sin b \cos a$ plus $\sin\pi = 0$, $\cos\pi = -1$.
$= e^{-\pi s}\dfrac{-1}{s^2+1}$	Basic Laplace table. Identity verified.

15 Example (Trigonometric formulas) Show the steps used to obtain these Laplace identities:

(a) $\mathbf{L}(t\cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$ (c) $\mathbf{L}(t^2\cos at) = \dfrac{2(s^3-3sa^2)}{(s^2+a^2)^3}$

(b) $\mathbf{L}(t\sin at) = \dfrac{2sa}{(s^2+a^2)^2}$ (d) $\mathbf{L}(t^2\sin at) = \dfrac{6s^2a-a^3}{(s^2+a^2)^3}$

Solution: The details for (a):

$$\begin{aligned}
\mathcal{L}(t\cos at) &= -(d/ds)\mathcal{L}(\cos at) && \text{Use } s\text{-differentiation.}\\
&= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) && \text{Basic Laplace table.}\\
&= \frac{s^2-a^2}{(s^2+a^2)^2} && \text{Calculus quotient rule.}
\end{aligned}$$

The details for (c):

$$\begin{aligned}
\mathcal{L}(t^2\cos at) &= -(d/ds)\mathcal{L}((-t)\cos at) && \text{Use } s\text{-differentiation.}\\
&= \frac{d}{ds}\left(-\frac{s^2-a^2}{(s^2+a^2)^2}\right) && \text{Result of (a).}\\
&= \frac{2s^3-6sa^2)}{(s^2+a^2)^3} && \text{Calculus quotient rule.}
\end{aligned}$$

The similar details for (b) and (d) are left as exercises.

16 **Example (Exponentials)** Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$ (c) $\mathcal{L}(te^{at}\cos bt) = \dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$

(b) $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ (d) $\mathcal{L}(te^{at}\sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Solution: Details for (a):

$$\begin{aligned}
\mathcal{L}(e^{at}\cos bt) &= \mathcal{L}(\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\
&= \left(\frac{s}{s^2+b^2}\right)\Bigg|_{s\to s-a} && \text{Basic Laplace table.}\\
&= \frac{s-a}{(s-a)^2+b^2} && \text{Verified (a).}
\end{aligned}$$

Details for (c):

$$\begin{aligned}
\mathcal{L}(te^{at}\cos bt) &= \mathcal{L}(t\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\
&= \left(-\frac{d}{ds}\mathcal{L}(\cos bt)\right)\Bigg|_{s\to s-a} && \text{Apply } s\text{-differentiation.}\\
&= \left(-\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)\right)\Bigg|_{s\to s-a} && \text{Basic Laplace table.}\\
&= \left(\frac{s^2-b^2}{(s^2+b^2)^2}\right)\Bigg|_{s\to s-a} && \text{Calculus quotient rule.}\\
&= \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2} && \text{Verified (c).}
\end{aligned}$$

Left as exercises are (b) and (d).

17 **Example (Hyperbolic functions)** Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

(a) $\mathbf{L}(\cosh at) = \dfrac{s}{s^2 - a^2}$ (c) $\mathbf{L}(t\cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(b) $\mathbf{L}(\sinh at) = \dfrac{a}{s^2 - a^2}$ (d) $\mathbf{L}(t\sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$

Solution: The details for (a):

$L(\cosh at) = \frac{1}{2}\left(L(e^{at}) + L(e^{-at})\right)$	Definition plus linearity of L.
$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$	Basic Laplace table.
$= \frac{s}{s^2 - a^2}$	Identity (a) verified.

The details for (d):

$L(t\sinh at) = -\frac{d}{ds}\left[\frac{a}{s^2 - a^2}\right]$	Apply the s-differentiation rule.
$= \frac{a(2s)}{(s^2 - a^2)^2}$	Calculus power rule; (d) verified.

Left as exercises are (b) and (c).

18 **Example (s-differentiation)** Solve $\mathbf{L}(f(t)) = \dfrac{2s}{(s^2 + 1)^2}$ for $f(t)$.

Solution: The solution is $f(t) = t\sin t$. The details:

$L(f(t)) = \frac{2s}{(s^2 + 1)^2}$	
$= -\frac{d}{ds}\left[\frac{1}{s^2 + 1}\right]$	Calculus power rule $(u^n)' = nu^{n-1}u'$.
$= -\frac{d}{ds}(L(\sin t))$	Basic Laplace table.
$= L(t\sin t)$	Apply the s-differentiation rule.
$f(t) = t\sin t$	Lerch's cancellation law.

19 **Example (First shift rule)** Solve $\mathbf{L}(f(t)) = \dfrac{s + 2}{2^2 + 2s + 2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-t}\cos t + e^{-t}\sin t$. The details:

$L(f(t)) = \frac{s + 2}{s^2 + 2s + 2}$	Signal for this method: the denominator has complex roots.
$= \frac{s + 2}{(s + 1)^2 + 1}$	Complete the square, denominator.

$$
\begin{aligned}
&= \frac{S+1}{S^2+1} && \text{Substitute } S \text{ for } s+1.\\
&= \frac{S}{S^2+1} + \frac{1}{S^2+1} && \text{Split into Laplace table entries.}\\
&= \mathcal{L}(\cos t) + \mathcal{L}(\sin t)|_{s\to S=s+1} && \text{Basic Laplace table.}\\
&= \mathcal{L}(e^{-t}\cos t) + \mathcal{L}(e^{-t}\sin t) && \text{First shift rule.}\\
f(t) &= e^{-t}\cos t + e^{-t}\sin t && \text{Invoke Lerch's cancellation law.}
\end{aligned}
$$

20 Example (Damped oscillator) **Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.**

Solution: The solution is $x(t) = e^{-t}\cos t$. The details:

$$
\begin{aligned}
&\mathcal{L}(x'') + 2\mathcal{L}(x') + 2\mathcal{L}(x) = \mathcal{L}(0) && \text{Apply } \mathcal{L} \text{ across the equation.}\\
&s\mathcal{L}(x') - x'(0) + 2\mathcal{L}(x') + 2\mathcal{L}(x) = 0 && \text{The } t\text{-derivative rule on } x'.\\
&s[s\mathcal{L}(x) - x(0)] - x'(0) && \text{The } t\text{-derivative rule on } x.\\
&+2[\mathcal{L}(x) - x(0)] + 2\mathcal{L}(x) = 0 &&\\
&(s^2+2s+2)\mathcal{L}(x) = 1 + s && \text{Use } x(0)=1,\ x'(0)=-1.\\
&\mathcal{L}(x) = \frac{s+1}{s^2+2s+2} && \text{Divide.}\\
&\quad = \frac{s+1}{(s+1)^2+1} && \text{Complete the square in the denominator.}\\
&\quad = \mathcal{L}(\cos t)|_{s\to s+1} && \text{Basic Laplace table.}\\
&\quad = \mathcal{L}(e^{-t}\cos t) && \text{First shifting rule.}\\
&x(t) = e^{-t}\cos t && \text{Invoke Lerch's cancellation law.}
\end{aligned}
$$

21 Example (Rectified sine wave) **Compute the Laplace transform of the rectified sine wave $f(t) = |\sin \omega t|$ and show it can be expressed in the form**

$$\mathcal{L}(|\sin \omega t|) = \frac{\omega \coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}.$$

Solution: The periodic function formula will be applied with period $P = 2\pi/\omega$. The calculation reduces to the evaluation of $J = \int_0^P f(t)e^{-st}dt$. Because $\sin \omega t \le 0$ on $\pi/\omega \le t \le 2\pi/\omega$, integral J can be written as $J = J_1 + J_2$, where

$$J_1 = \int_0^{\pi/\omega} \sin \omega t e^{-st}dt, \quad J_2 = \int_{\pi/\omega}^{2\pi/\omega} -\sin \omega t e^{-st}dt.$$

Integral tables give the result

$$\int \sin \omega t e^{-st}dt = -\frac{\omega e^{-st}\cos(\omega t)}{s^2+\omega^2} - \frac{s e^{-st}\sin(\omega t)}{s^2+\omega^2}.$$

Then

$$J_1 = \frac{\omega(e^{-\pi * s/\omega}+1)}{s^2+\omega^2}, \quad J_2 = \frac{\omega(e^{-2\pi s/\omega}+e^{-\pi s/\omega})}{s^2+\omega^2},$$

$$J = \frac{\omega(e^{-\pi s/\omega}+1)^2}{s^2+\omega^2}.$$

The remaining challenge is to write the answer for $\mathcal{L}(f(t))$ in terms of coth. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{J}{1-e^{-Ps}} && \text{Periodic function formula.}\\
&= \frac{J}{(1-e^{-Ps/2})(1+e^{-Ps/2})} && \text{Apply } 1-x^2=(1-x)(1+x),\ x=e^{-Ps/2}.\\
&= \frac{\omega(1+e^{-Ps/2})}{(1-e^{-Ps/2})(s^2+\omega^2)} && \text{Cancel factor } 1+e^{-Ps/2}.\\
&= \frac{e^{Ps/4}+e^{-Ps/4}}{e^{Ps/4}-e^{-Ps/4}}\frac{\omega}{s^2+\omega^2} && \text{Factor out } e^{-Ps/4}\text{, then cancel.}\\
&= \frac{2\cosh(Ps/4)}{2\sinh(Ps/4)}\frac{\omega}{s^2+\omega^2} && \text{Apply cosh, sinh identities.}\\
&= \frac{\omega\coth(Ps/4)}{s^2+\omega^2} && \text{Use } \coth u=\cosh u/\sinh u.\\
&= \frac{\omega\coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2} && \text{Identity verified.}
\end{aligned}$$

22 Example (Half–wave rectification) **Compute the Laplace transform of the half–wave rectification of $\sin\omega t$, denoted $g(t)$, in which the negative cycles of $\sin\omega t$ have been canceled to create $g(t)$. Show in particular that**

$$\mathcal{L}(g(t)) = \frac{1}{2}\frac{\omega}{s^2+\omega^2}\left(1+\coth\left(\frac{\pi s}{2\omega}\right)\right)$$

Solution: The half-wave rectification of $\sin\omega t$ is $g(t)=(\sin\omega t+|\sin\omega t|)/2$. Therefore, the basic Laplace table plus the result of Example 21 give

$$\begin{aligned}
\mathcal{L}(2g(t)) &= \mathcal{L}(\sin\omega t)+\mathcal{L}(|\sin\omega t|)\\
&= \frac{\omega}{s^2+\omega^2}+\frac{\omega\cosh(\pi s/(2\omega))}{s^2+\omega^2}\\
&= \frac{\omega}{s^2+\omega^2}(1+\cosh(\pi s/(2\omega)))
\end{aligned}$$

Dividing by 2 produces the identity.

23 Example (Shifting rules) **Solve $\mathcal{L}(f(t)) = e^{-3s}\dfrac{s+1}{s^2+2s+2}$ for $f(t)$.**

Solution: The answer is $f(t)=e^{3-t}\cos(t-3)H(t-3)$. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= e^{-3s}\frac{s+1}{(s+1)^2+1} && \text{Complete the square.}\\
&= e^{-3s}\frac{S}{S^2+1} && \text{Replace } s+1 \text{ by } S.\\
&= e^{-3S+3}\left(\mathcal{L}(\cos t)\right)\big|_{s\to S=s+1} && \text{Basic Laplace table.}
\end{aligned}$$

$$= e^3 \cdot \left(e^{-3s}\mathcal{L}(\cos t)\right)\Big|_{s\to S=s+1} \qquad \text{Regroup factor } e^{-3S}.$$
$$= e^3 \left(\mathcal{L}(\cos(t-3)H(t-3))\right)\Big|_{s\to S=s+1} \qquad \text{Second shifting rule.}$$
$$= e^3\mathcal{L}(e^{-t}\cos(t-3)H(t-3)) \qquad \text{First shifting rule.}$$
$$f(t) = e^{3-t}\cos(t-3)H(t-3) \qquad \text{Lerch's cancellation law.}$$

24 Example () Solve $\mathbf{L}(f(t)) = \dfrac{s+7}{s^2+4s+8}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$. The details:

$$\mathcal{L}(f(t)) = \frac{s+7}{(s+2)^2+4} \qquad \text{Complete the square.}$$
$$= \frac{S+5}{S^2+4} \qquad \text{Replace } s+2 \text{ by } S.$$
$$= \frac{S}{S^2+4} + \frac{5}{2}\frac{2}{S^2+4} \qquad \text{Split into table entries.}$$
$$= \left(\frac{s}{s^2+4} + \frac{5}{2}\frac{2}{s^2+4}\right)\Big|_{s\to S=s+2} \qquad \text{Prepare for shifting rule.}$$
$$= \left(\mathcal{L}(\cos 2t) + \frac{5}{2}\mathcal{L}(\sin 2t)\right)\Big|_{s\to S=s+2} \qquad \text{Basic Laplace table.}$$
$$= \mathcal{L}(e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t)) \qquad \text{First shifting rule.}$$
$$f(t) = e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t) \qquad \text{Lerch's cancellation law.}$$

Heaviside's Method

This practical method was popularized by the English electrical engineer Oliver Heaviside (1850–1925). A typical application of the method is to solve

$$\frac{2s}{(s+1)(s^2+1)} = \mathbf{L}(f(t))$$

for the t-expression $f(t) = -e^{-t} + \cos t + \sin t$. The details in Heaviside's method involve a sequence of easy-to-learn college algebra steps.

More precisely, Heaviside's method systematically converts a polynomial quotient

$$(1)\qquad \frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m}$$

into the form $\mathbf{L}(f(t))$ for some expression $f(t)$. It is assumed that $a_0, .., a_n, b_0, \ldots, b_m$ are constants and the polynomial quotient (1) has limit zero at $s = \infty$.

Partial Fraction Theory

In college algebra, it is shown that a rational function (1) can be expressed as the sum of terms of the form

$$(2)\qquad \frac{A}{(s - s_0)^k}$$

where A is a real or complex constant and $(s - s_0)^k$ divides the denominator in (1). In particular, s_0 is a *root* of the denominator in (1).

Assume fraction (1) has real coefficients. If s_0 in (2) is real, then A is *real*. If $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also appears, where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding terms in (2) turn out to be complex conjugates of one another, which can be combined in terms of *real* numbers B and C as

$$(3)\qquad \frac{A}{(s - s_0)^k} + \frac{\overline{A}}{(s - \overline{s_0})^k} = \frac{B + Cs}{((s-\alpha)^2 + \beta^2)^k}.$$

Simple Roots. Assume that (1) has *real coefficients* and the denominator of the fraction (1) has distinct real roots $s_1, \ldots, s_N$ and distinct complex roots $\alpha_1 + i\beta_1, \ldots, \alpha_M + i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$(4)\qquad \frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s - \alpha_q)}{(s - \alpha_q)^2 + \beta_q^2}.$$

Multiple Roots. Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly multiple roots. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$, $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$(5)\qquad \sum_{p=1}^{N}\sum_{1\le k\le N_p}\frac{A_{p,k}}{(s-s_p)^k}+\sum_{q=1}^{M}\sum_{1\le k\le M_q}\frac{B_{q,k}+C_{q,k}(s-\alpha_q)}{((s-\alpha_q)^2+\beta_q^2)^k}.$$

Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made; see page 266.

To illustrate Oliver Heaviside's ideas, consider the problem details

$$(6)\qquad \begin{aligned}\frac{2s+1}{s(s-1)(s+1)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1}\\ &= \mathbf{L}(A)+\mathbf{L}(Be^{t})+\mathbf{L}(Ce^{-t})\\ &= \mathbf{L}(A+Be^{t}+Ce^{-t})\end{aligned}$$

The first line (6) uses college algebra partial fractions. The second and third lines use the Laplace integral table and properties of $\mathbf{L}$.

Heaviside's mysterious method. Oliver Heaviside proposed to find in (6) the constant $C = \frac{1}{2}$ by a cover − up method:

$$\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.$$

The *instructions* are to cover–up the matching factors $(s + 1)$ on the left and right with box $\boxed{}$, then evaluate on the left at the *root* s which makes the contents of the box zero. The other terms on the right are replaced by zero.

To justify Heaviside's cover–up method, multiply (6) by the denominator $s + 1$ of partial fraction $C/(s + 1)$:

$$\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}}=\frac{A\boxed{(s+1)}}{s}+\frac{B\boxed{(s+1)}}{s-1}+\frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.$$

Set $\boxed{(s+1)} = 0$ in the display. Cancellations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$\left.\frac{2s+1}{s(s-1)}\right|_{s+1=0} = C.$$

The factor $(s+1)$ in (6) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Extension to Multiple Roots. An extension of Heaviside's method is possible for the case of repeated roots. The basic idea is to *factor–out the repeats*. To illustrate, consider the partial fraction expansion details

$R = \dfrac{1}{(s+1)^2(s+2)}$	A sample rational function having repeated roots.
$= \dfrac{1}{s+1}\left[\dfrac{1}{(s+1)(s+2)}\right]$	Factor–out the repeats.
$= \dfrac{1}{s+1}\left[\dfrac{1}{s+1} + \dfrac{-1}{s+2}\right]$	Apply the cover–up method to the simple root fraction.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{(s+1)(s+2)}$	Multiply.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{s+1} + \dfrac{1}{s+2}$	Apply the cover–up method to the last fraction on the right.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods. Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$R = \dfrac{1}{(s+1)^2(s+2)}$	A fraction with multiple roots.
$= \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$	See equation (5).
$= \dfrac{A}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	Find B and C by Heaviside's cover–up method.
$= \dfrac{-1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	Multiply by $s+1$. Set $s=\infty$. Then $0 = A + 1$.

The illustration works for one root of multiplicity two, because $s=\infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.

7.5 Heaviside Step and Dirac Delta

Heaviside Function. The unit step function or Heaviside function is defined by

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The most often–used formula involving the Heaviside function is the characteristic function of the interval $a \leq t < b$, given by

$$(1) \qquad H(t-a) - H(t-b) = \begin{cases} 1 & a \leq t < b, \\ 0 & t < a, \quad t \geq b. \end{cases}$$

To illustrate, a square wave $\text{sqw}(t) = (-1)^{\text{floor}(t)}$ can be written in the series form

$$\sum_{n=0}^{\infty} (-1)^n (H(t-n) - H(t-n-1)).$$

Dirac Delta. A precise mathematical definition of the Dirac delta, denoted δ, is not possible to give here. Following its inventor P. Dirac, the definition should be

$$\delta(t) = dH(t).$$

The latter is nonsensical, because the unit step does not have a calculus derivative at $t = 0$. However, $dH(t)$ could have the meaning of a Riemann-Stieltjes integrator, which restrains $dH(t)$ to have meaning only under an integral sign. It is in this sense that the Dirac delta δ is defined.

What do we mean by the differential equation

$$x'' + 16x = 5\delta(t - t_0)?$$

The equation $x'' + 16x = f(t)$ represents a spring-mass system without damping having Hooke's constant 16, subject to external force $f(t)$. In a mechanical context, the Dirac delta term $5\delta(t - t_0)$ is an *idealization* of a hammer-hit at time $t = t_0 > 0$ with impulse 5.

More precisely, the forcing term $f(t)$ can be formally written as a Riemann-Stieltjes integrator $5dH(t-t_0)$ where H is Heaviside's unit step function. The Dirac delta or "derivative of the Heaviside unit step," nonsensical as it may appear, is realized in applications via the two-sided or central difference quotient

$$\frac{H(t+h) - H(t-h)}{2h} \approx dH(t).$$

Therefore, the force $f(t)$ in the idealization $5\delta(t - t_0)$ is given for $h > 0$ very small by the approximation

$$f(t) \approx 5\,\frac{H(t - t_0 + h) - H(t - t_0 - h)}{2h}.$$

The *impulse*[2] of the approximated force over a large interval $[a, b]$ is computed from

$$\int_a^b f(t)\,dt \approx 5\int_{-h}^{h} \frac{H(t - t_0 + h) - H(t - t_0 - h)}{2h}\,dt = 5,$$

due to the integrand being $1/(2h)$ on $|t - t_0| < h$ and otherwise 0.

Modeling Impulses. One argument for the Dirac delta idealization is that an infinity of choices exist for modeling an impulse. There are in addition to the central difference quotient two other popular difference quotients, the forward quotient $(H(t + h) - H(t))/h$ and the backward quotient $(H(t) - H(t - h))/h$ ($h > 0$ assumed). In reality, h is unknown in any application, and the impulsive force of a hammer hit is hardly constant, as is supposed by this naive modeling.

The modeling logic often applied for the Dirac delta is that the external force $f(t)$ is used in the model in a limited manner, in which only the momentum $p = mv$ is important. More precisely, only the change in momentum or impulse is important, $\int_a^b f(t)\,dt = \Delta p = mv(b) - mv(a)$.

The precise force $f(t)$ is replaced during the modeling by a simplistic piecewise-defined force that has exactly the same impulse Δp. The replacement is justified by arguing that if only the impulse is important, and not the actual details of the force, then both models should give similar results.

Function or Operator? The work of physics Nobel prize winner P. Dirac (1902–1984) proceeded for about 20 years before the mathematical community developed a sound mathematical theory for his impulsive force representations. A systematic theory was developed in 1936 by the soviet mathematician S. Sobolev. The French mathematician L. Schwartz further developed the theory in 1945. He observed that the idealization is not a function but an operator or *linear functional*, in particular, δ maps or *associates* to each function $\varphi(t)$ its value at $t = 0$, in short, $\delta(\varphi) = \varphi(0)$. This fact was observed early on by Dirac and others, during the replacement of simplistic forces by δ. In Laplace theory, there is a natural encounter with the ideas, because $\mathbf{L}(f(t))$ routinely appears on the right of the equation after transformation. This term, in the case

[2]Momentum is defined to be mass times velocity. If the force f is given by Newton's law as $f(t) = \frac{d}{dt}(mv(t))$ and $v(t)$ is velocity, then $\int_a^b f(t)\,dt = mv(b) - mv(a)$ is the net momentum or impulse.

of an impulsive force $f(t) = c(H(t - t_0 - h) - H(t - t_0 + h))/(2h)$, evaluates for $t_0 > 0$ and $t_0 - h > 0$ as follows:

$$\begin{aligned}
\mathbf{L}(f(t)) &= \int_0^\infty \frac{c}{2h}(H(t - t_0 - h) - H(t - t_0 + h))e^{-st}dt \\
&= \int_{t_0-h}^{t_0+h} \frac{c}{2h} e^{-st}dt \\
&= ce^{-st_0}\left[\frac{e^{sh} - e^{-sh}}{2sh}\right]
\end{aligned}$$

The factor $\dfrac{e^{sh} - e^{-sh}}{2sh}$ is approximately 1 for $h > 0$ small, because of L'Hospital's rule. The immediate conclusion is that we should replace the impulsive force f by an equivalent one f^* such that

$$\mathbf{L}(f^*(t)) = ce^{-st_0}.$$

Well, *there is no such function f^**!

The apparent mathematical flaw in this idea was resolved by the work of L. Schwartz on distributions. In short, there is a solid foundation for introducing f^*, but unfortunately the mathematics involved is not elementary nor especially accessible to those readers whose background is just calculus.

Practising engineers and scientists might be able to ignore the vast literature on distributions, citing the example of physicist P. Dirac, who succeeded in applying impulsive force ideas without the distribution theory developed by S. Sobolev and L. Schwartz. This will not be the case for those who wish to read current literature on partial differential equations, because the work on distributions has forever changed the required background for that topic.

Laplace Table Derivations

Verified here are two Laplace tables, the minimal Laplace Table 7.2-4 and its extension Table 7.2-5. Largely, this section is for reading, as it is designed to enrich lectures and to aid readers who study alone.

Derivation of Laplace integral formulas in Table 7.2-4, page 252.

- Proof of $\mathcal{L}(t^n) = n!/s^{1+n}$:

The first step is to evaluate $\mathcal{L}(t^n)$ for $n = 0$.

$$\begin{aligned}
\mathcal{L}(1) &= \int_0^\infty (1)e^{-st}\,dt && \text{Laplace integral of } f(t) = 1.\\
&= -(1/s)e^{-st}\big|_{t=0}^{t=\infty} && \text{Evaluate the integral.}\\
&= 1/s && \text{Assumed } s > 0 \text{ to evaluate } \lim_{t\to\infty} e^{-st}.
\end{aligned}$$

The value of $\mathcal{L}(t^n)$ for $n = 1$ can be obtained by s-differentiation of the relation $\mathcal{L}(1) = 1/s$, as follows.

$$\begin{aligned}
\frac{d}{ds}\mathcal{L}(1) &= \frac{d}{ds}\int_0^\infty (1)e^{-st}\,dt && \text{Laplace integral for } f(t) = 1.\\
&= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)dt && \text{Used } \frac{d}{ds}\int_a^b F\,dt = \int_a^b \frac{dF}{ds}\,dt.\\
&= \int_0^\infty (-t)e^{-st}\,dt && \text{Calculus rule } (e^u)' = u'e^u.\\
&= -\mathcal{L}(t) && \text{Definition of } \mathcal{L}(t).
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}(t) &= -\frac{d}{ds}\mathcal{L}(1) && \text{Rewrite last display.}\\
&= -\frac{d}{ds}(1/s) && \text{Use } \mathcal{L}(1) = 1/s.\\
&= 1/s^2 && \text{Differentiate.}
\end{aligned}$$

This idea can be repeated to give $\mathcal{L}(t^2) = -\frac{d}{ds}\mathcal{L}(t)$ and hence $\mathcal{L}(t^2) = 2/s^3$. The pattern is $\mathcal{L}(t^n) = -\frac{d}{ds}\mathcal{L}(t^{n-1})$ which gives $\mathcal{L}(t^n) = n!/s^{1+n}$.

- Proof of $\mathcal{L}(e^{at}) = 1/(s-a)$:

The result follows from $\mathcal{L}(1) = 1/s$, as follows.

$$\begin{aligned}
\mathcal{L}(e^{at}) &= \int_0^\infty e^{at}e^{-st}\,dt && \text{Direct Laplace transform.}\\
&= \int_0^\infty e^{-(s-a)t}\,dt && \text{Use } e^Ae^B = e^{A+B}.\\
&= \int_0^\infty e^{-St}\,dt && \text{Substitute } S = s-a.\\
&= 1/S && \text{Apply } \mathcal{L}(1) = 1/s.\\
&= 1/(s-a) && \text{Back-substitute } S = s-a.
\end{aligned}$$

- Proof of $\mathcal{L}(\cos bt) = s/(s^s + b^2)$ and $\mathcal{L}(\sin bt) = b/(s^s + b^2)$:

Use will be made of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, usually first introduced in trigonometry. In this formula, θ is a real number (in radians) and $i = \sqrt{-1}$ is the complex unit.

$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$	Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st}.
$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Integrate $t = 0$ to $t = \infty$. Use properties of integrals.
$\frac{1}{s - ib} = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Evaluate the left side using $\mathcal{L}(e^{at}) = 1/(s-a)$, $a = ib$.
$\frac{1}{s - ib} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Direct Laplace transform definition.
$\frac{s + ib}{s^2 + b^2} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Use complex rule $1/z = \bar{z}/\lvert z\rvert^2$, $z = A + iB$, $\bar{z} = A - iB$, $\lvert z\rvert = \sqrt{A^2 + B^2}$.
$\frac{s}{s^2 + b^2} = \mathcal{L}(\cos bt)$	Extract the real part.
$\frac{b}{s^2 + b^2} = \mathcal{L}(\sin bt)$	Extract the imaginary part.

Derivation of Laplace integral formulas in Table 7.2-5, page 252.

- Proof of the Heaviside formula $\mathcal{L}(H(t-a)) = e^{-as}/s$.

$\mathcal{L}(H(t-a)) = \int_0^\infty H(t-a)e^{-st}dt$	Direct Laplace transform. Assume $a \geq 0$.
$= \int_a^\infty (1)e^{-st}dt$	Because $H(t-a) = 0$ for $0 \leq t < a$.
$= \int_0^\infty (1)e^{-s(x+a)}dx$	Change variables $t = x + a$.
$= e^{-as}\int_0^\infty (1)e^{-sx}dx$	Constant e^{-as} moves outside integral.
$= e^{-as}(1/s)$	Apply $\mathcal{L}(1) = 1/s$.

- Proof of the Dirac delta formula $\mathcal{L}(\delta(t-a)) = e^{-as}$.

The ***definition*** of the delta function is a formal one, in which every occurrence of $\delta(t-a)dt$ under an integrand is replaced by $dH(t-a)$. The differential symbol $dH(t-a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in [?] for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N\to\infty}\sum_{n=1}^N f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \to \infty$.

The steps in computing the Laplace integral of the delta function appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$.

$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt$	Laplace integral, $a > 0$ assumed.
$= \int_0^\infty e^{-st}dH(t-a)$	Replace $\delta(t-a)dt$ by $dH(t-a)$.
$= \lim_{M\to\infty}\int_0^M e^{-st}dH(t-a)$	Definition of improper integral.

$$= e^{-sa} \qquad \text{Explained below.}$$

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st}dH(t-a) = \lim_{N\to\infty}\sum_{n=0}^{N-1} e^{-st_n}\left(H(t_n - a) - H(t_{n-1} - a)\right)$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1\le n\le N}(t_n - t_{n-1})$ approaches zero as $N \to \infty$. Given a partition, if $t_{n-1} < a \le t_n$, then $H(t_n-a)-H(t_{n-1}-a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term e^{-st_n}. This term approaches e^{-sa} as $N \to \infty$, because t_n must approach a.

- Proof of $\mathcal{L}(\text{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$:

The library function floor present in computer languages C and Fortran is defined by $\text{floor}(x)$ = greatest whole integer $\le x$, e.g., $\text{floor}(5.2) = 5$ and $\text{floor}(-1.9) = -2$. The computation of the Laplace integral of $\text{floor}(t)$ requires ideas from infinite series, as follows.

$$\begin{aligned}
F(s) &= \int_0^\infty \text{floor}(t)e^{-st}dt && \text{Laplace integral definition.}\\
&= \sum_{n=0}^\infty \int_n^{n+1}(n)e^{-st}dt && \text{On } n \le t < n+1,\ \text{floor}(t) = n.\\
&= \sum_{n=0}^\infty \frac{n}{s}\left(e^{-ns} - e^{-ns-s}\right) && \text{Evaluate each integral.}\\
&= \frac{1-e^{-s}}{s}\sum_{n=0}^\infty ne^{-sn} && \text{Common factor removed.}\\
&= \frac{x(1-x)}{s}\sum_{n=0}^\infty nx^{n-1} && \text{Define } x = e^{-s}.\\
&= \frac{x(1-x)}{s}\frac{d}{dx}\sum_{n=0}^\infty x^n && \text{Term-by-term differentiation.}\\
&= \frac{x(1-x)}{s}\frac{d}{dx}\frac{1}{1-x} && \text{Geometric series sum.}\\
&= \frac{x}{s(1-x)} && \text{Compute the derivative, simplify.}\\
&= \frac{e^{-s}}{s(1-e^{-s})} && \text{Substitute } x = e^{-s}.
\end{aligned}$$

To evaluate the Laplace integral of $\text{floor}(t/a)$, a change of variables is made.

$$\begin{aligned}
\mathcal{L}(\text{floor}(t/a)) &= \int_0^\infty \text{floor}(t/a)e^{-st}dt && \text{Laplace integral definition.}\\
&= a\int_0^\infty \text{floor}(r)e^{-asr}dr && \text{Change variables } t = ar.\\
&= aF(as) && \text{Apply the formula for } F(s).\\
&= \frac{e^{-as}}{s(1-e^{-as})} && \text{Simplify.}
\end{aligned}$$

- Proof of $\mathcal{L}(\text{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$:

The square wave defined by $\text{sqw}(x) = (-1)^{\text{floor}(x)}$ is periodic of period 2 and piecewise-defined. Let $P = \int_0^2 \text{sqw}(t)e^{-st}dt$.

$$
\begin{aligned}
P &= \int_0^1 \text{sqw}(t)e^{-st}\,dt + \int_1^2 \text{sqw}(t)e^{-st}\,dt && \text{Apply } \int_a^b = \int_a^c + \int_c^b. \\
&= \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt && \text{Use sqw}(x) = 1 \text{ on } 0 \le x < 1 \text{ and sqw}(x) = -1 \text{ on } 1 \le x < 2. \\
&= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) && \text{Evaluate each integral.} \\
&= \frac{1}{s}(1 - e^{-s})^2 && \text{Collect terms.}
\end{aligned}
$$

An intermediate step is to compute the Laplace integral of $\text{sqw}(t)$:

$$
\begin{aligned}
L(\text{sqw}(t)) &= \frac{\int_0^2 \text{sqw}(t)e^{-st}\,dt}{1 - e^{-2s}} && \text{Periodic function formula, page 275.} \\
&= \frac{1}{s}(1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}. && \text{Use the computation of } P \text{ above.} \\
&= \frac{1}{s}\frac{1 - e^{-s}}{1 + e^{-s}}. && \text{Factor } 1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s}). \\
&= \frac{1}{s}\frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}. && \text{Multiply the fraction by } e^{s/2}/e^{s/2}. \\
&= \frac{1}{s}\frac{\sinh(s/2)}{\cosh(s/2)}. && \text{Use } \sinh u = (e^u - e^{-u})/2, \\
& && \cosh u = (e^u + e^{-u})/2. \\
&= \frac{1}{s}\tanh(s/2). && \text{Use } \tanh u = \sinh u/\cosh u.
\end{aligned}
$$

To complete the computation of $L(\text{sqw}(t/a))$, a change of variables is made:

$$
\begin{aligned}
L(\text{sqw}(t/a)) &= \int_0^\infty \text{sqw}(t/a)e^{-st}\,dt && \text{Direct transform.} \\
&= \int_0^\infty \text{sqw}(r)e^{-asr}(a)\,dr && \text{Change variables } r = t/a. \\
&= \frac{a}{as}\tanh(as/2) && \text{See } L(\text{sqw}(t)) \text{ above.} \\
&= \frac{1}{s}\tanh(as/2)
\end{aligned}
$$

- Proof of $L(a\,\text{trw}(t/a)) = \frac{1}{s^2}\tanh(as/2)$:

The triangular wave is defined by $\text{trw}(t) = \int_0^t \text{sqw}(x)dx$.

$$
\begin{aligned}
L(a\,\text{trw}(t/a)) &= \frac{1}{s}(f(0) + L(f'(t))) && \text{Let } f(t) = a\,\text{trw}(t/a). \text{ Use } L(f'(t)) = sL(f(t)) - f(0), \text{ page 251.} \\
&= \frac{1}{s}L(\text{sqw}(t/a)) && \text{Use } f(0) = 0, \left(a\int_0^{t/a} \text{sqw}(x)dx\right)' = \text{sqw}(t/a). \\
&= \frac{1}{s^2}\tanh(as/2) && \text{Table entry for sqw.}
\end{aligned}
$$

- Proof of $L(t^a) = \dfrac{\Gamma(1 + a)}{s^{1+a}}$:

$$
\begin{aligned}
L(t^a) &= \int_0^\infty t^a e^{-st}\,dt && \text{Direct Laplace transform.} \\
&= \int_0^\infty (u/s)^a e^{-u}\,du/s && \text{Change variables } u = st,\ du = s\,dt.
\end{aligned}
$$

$$= \frac{1}{s^{1+a}} \int_0^\infty u^a e^{-u}\, du$$

$$= \frac{1}{s^{1+a}} \Gamma(1+a).$$ Where $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$, by definition.

The ***generalized factorial function*** $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2)\cdots(n)$ in case $x = n + 1$ is an integer. In literature, $a!$ means $\Gamma(1 + a)$. For more details about the Gamma function, see Abramowitz and Stegun [?], or `maple` documentation.

- Proof of $\mathcal{L}(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$:

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(1+(-1/2))}{s^{1-1/2}}$$ Apply the previous formula.

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$ Use $\Gamma(1/2) = \sqrt{\pi}$.

7.7 Transform Properties

Collected here are the major theorems and their proofs for the manipulation of Laplace transform tables.

Theorem 4 (Linearity)
The Laplace transform has these inherited integral properties:

$$\text{(a)}\quad \mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)),$$
$$\text{(b)}\quad \mathcal{L}(cf(t)) = c\mathcal{L}(f(t)).$$

Theorem 5 (The t-Derivative Rule)
Let $y(t)$ be continuous, of exponential order and let $f'(t)$ be piecewise continuous on $t \ge 0$. Then $\mathcal{L}(y'(t))$ exists and

$$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0).$$

Theorem 6 (The t-Integral Rule)
Let $g(t)$ be of exponential order and continuous for $t \ge 0$. Then

$$\mathcal{L}\left(\int_0^t g(x)\,dx\right) = \frac{1}{s}\mathcal{L}(g(t)).$$

Theorem 7 (The s-Differentiation Rule)
Let $f(t)$ be of exponential order. Then

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t)).$$

Theorem 8 (First Shifting Rule)
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\big|_{s\to(s-a)}.$$

Theorem 9 (Second Shifting Rule)
Let $f(t)$ and $g(t)$ be of exponential order and assume $a \ge 0$. Then

$$\text{(a)}\quad \mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t)),$$
$$\text{(b)}\quad \mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a)).$$

Theorem 10 (Periodic Function Rule)
Let $f(t)$ be of exponential order and satisfy $f(t+P) = f(t)$. Then

$$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}\,dt}{1-e^{-Ps}}.$$

Theorem 11 (Convolution Rule)
Let $f(t)$ and $g(t)$ be of exponential order. Then

$$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)g(t-x)\,dx\right).$$

Proof of Theorem 4 (linearity):

$$
\begin{aligned}
\text{LHS} &= \mathcal{L}(f(t)+g(t)) && \text{Left side of the identity in (a).}\\
&= \int_0^\infty (f(t)+g(t))e^{-st}\,dt && \text{Direct transform.}\\
&= \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt && \text{Calculus integral rule.}\\
&= \mathcal{L}(f(t)) + \mathcal{L}(g(t)) && \text{Equals RHS; identity (a) verified.}\\
\text{LHS} &= \mathcal{L}(cf(t)) && \text{Left side of the identity in (b).}\\
&= \int_0^\infty cf(t)e^{-st}\,dt && \text{Direct transform.}\\
&= c\int_0^\infty f(t)e^{-st}\,dt && \text{Calculus integral rule.}\\
&= c\mathcal{L}(f(t)) && \text{Equals RHS; identity (b) verified.}
\end{aligned}
$$

Proof of Theorem 5 (t-derivative rule): Already $\mathcal{L}(f(t))$ exists, because f is of exponential order and continuous. On an interval $[a,b]$ where f' is continuous, integration by parts using $u = e^{-st}$, $dv = f'(t)dt$ gives

$$
\begin{aligned}
\int_a^b f'(t)e^{-st}\,dt &= f(t)e^{-st}\Big|_{t=a}^{t=b} - \int_a^b f(t)(-s)e^{-st}\,dt\\
&= -f(a)e^{-sa} + f(b)e^{-sb} + s\int_a^b f(t)e^{-st}\,dt.
\end{aligned}
$$

On any interval $[0,N]$, there are finitely many intervals $[a,b]$ on each of which f' is continuous. Add the above equality across these finitely many intervals $[a,b]$. The boundary values on adjacent intervals match and the integrals add to give

$$\int_0^N f'(t)e^{-st}\,dt = -f(0)e^0 + f(N)e^{-sN} + s\int_0^N f(t)e^{-st}\,dt.$$

Take the limit across this equality as $N \to \infty$. Then the right side has limit $-f(0) + s\mathcal{L}(f(t))$, because of the existence of $\mathcal{L}(f(t))$ and $\lim_{t\to\infty} f(t)e^{-st} = 0$ for large s. Therefore, the left side has a limit, and by definition $\mathcal{L}(f'(t))$ exists and $\mathcal{L}(f'(t)) = -f(0) + s\mathcal{L}(f(t))$.

Proof of Theorem 6 (t-Integral rule): Let $f(t) = \int_0^t g(x)dx$. Then f is of exponential order and continuous. The details:

$$
\begin{aligned}
\mathcal{L}\left(\int_0^t g(x)dx\right) &= \mathcal{L}(f(t)) && \text{By definition.}\\
&= \frac{1}{s}\mathcal{L}(f'(t)) && \text{Because } f(0)=0 \text{ implies } \mathcal{L}(f'(t)) = s\mathcal{L}(f(t)).\\
&= \frac{1}{s}\mathcal{L}(g(t)) && \text{Because } f' = g \text{ by the Fundamental theorem of calculus.}
\end{aligned}
$$

Proof of Theorem 7 (s-differentiation): We prove the equivalent relation $\mathcal{L}((-t)f(t)) = (d/ds)\mathcal{L}(f(t))$. If f is of exponential order, then so is $(-t)f(t)$, therefore $\mathcal{L}((-t)f(t))$ exists. It remains to show the s-derivative exists and satisfies the given equality.

The proof below is based in part upon the calculus inequality

$$(1)\qquad \left|e^{-x} + x - 1\right| \le x^2, \quad x \ge 0.$$

The inequality is obtained from two applications of the ***mean value theorem*** $g(b)-g(a) = g'(\bar{x})(b-a)$, which gives $e^{-x}+x-1 = x\bar{x}e^{-x_1}$ with $0 \le x_1 \le \bar{x} \le x$. In addition, the existence of $\mathcal{L}(t^2|f(t)|)$ is used to define $s_0 > 0$ such that $\mathcal{L}(t^2|f(t)|) \le 1$ for $s > s_0$. This follows from the transform existence theorem for functions of exponential order, where it is shown that the transform has limit zero at $s = \infty$.

Consider $h \ne 0$ and the Newton quotient $Q(s,h) = (F(s+h) - F(s))/h$ for the s-derivative of the Laplace integral. We have to show that

$$\lim_{h\to 0} |Q(s,h) - \mathcal{L}((-t)f(t))| = 0.$$

This will be accomplished by proving for $s > s_0$ and $s + h > s_0$ the inequality

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h|.$$

For $h \ne 0$,

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)\frac{e^{-st-ht} - e^{-st} + the^{-st}}{h}\,dt.$$

Assume $h > 0$. Due to the exponential rule $e^{A+B} = e^Ae^B$, the quotient in the integrand simplifies to give

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)e^{-st}\left(\frac{e^{-ht} + th - 1}{h}\right)dt.$$

Inequality (1) applies with $x = ht \ge 0$, giving

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h| \int_0^\infty t^2|f(t)|e^{-st}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$ which for $s > s_0$ is bounded by $|h|$, completing the proof for $h > 0$. If $h < 0$, then a similar calculation is made to obtain

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h| \int_0^\infty t^2|f(t)|e^{-st-ht}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$ evaluated at $s + h$ instead of s. If $s + h > s_0$, then the right side is bounded by $|h|$, completing the proof for $h < 0$.

Proof of Theorem 8 (first shifting rule): The left side LHS of the equality can be written because of the exponential rule $e^Ae^B = e^{A+B}$ as

$$\text{LHS} = \int_0^\infty f(t)e^{-(s-a)t}dt.$$

This integral is $\mathcal{L}(f(t))$ with s replaced by $s - a$, which is precisely the meaning of the right side RHS of the equality. Therefore, LHS = RHS.

Proof of Theorem 9 (second shifting rule): The details for (a) are

$$\begin{aligned}\text{LHS} &= \mathcal{L}(H(t-a)f(t-a)) \\ &= \int_0^\infty H(t-a)f(t-a)e^{-st}dt \quad \text{Direct transform.}\end{aligned}$$

$$\begin{aligned}
&= \int_a^\infty H(t-a)f(t-a)e^{-st}dt && \text{Because } a \ge 0 \text{ and } H(x)=0 \text{ for } x<0.\\
&= \int_0^\infty H(x)f(x)e^{-s(x+a)}dx && \text{Change variables } x=t-a,\ dx=dt.\\
&= e^{-sa}\int_0^\infty f(x)e^{-sx}dx && \text{Use } H(x)=1 \text{ for } x\ge 0.\\
&= e^{-sa}\mathcal{L}(f(t)) && \text{Direct transform.}\\
&= \text{RHS} && \text{Identity (a) verified.}
\end{aligned}$$

In the details for (b), let $f(t) = g(t+a)$, then

$$\begin{aligned}
\text{LHS} &= \mathcal{L}(H(t-a)g(t))\\
&= \mathcal{L}(H(t-a)f(t-a)) && \text{Use } f(t-a)=g(t-a+a)=g(t).\\
&= e^{-sa}\mathcal{L}(f(t)) && \text{Apply (a).}\\
&= e^{-sa}\mathcal{L}(g(t+a)) && \text{Because } f(t)=g(t+a).\\
&= \text{RHS} && \text{Identity (b) verified.}
\end{aligned}$$

Proof of Theorem 10 (periodic function rule):

$$\begin{aligned}
\text{LHS} &= \mathcal{L}(f(t))\\
&= \int_0^\infty f(t)e^{-st}dt && \text{Direct transform.}\\
&= \sum_{n=0}^\infty \int_{nP}^{nP+P} f(t)e^{-st}dt && \text{Additivity of the integral.}\\
&= \sum_{n=0}^\infty \int_0^P f(x+nP)e^{-sx-nPs}dx && \text{Change variables } t=x+nP.\\
&= \sum_{n=0}^\infty e^{-nPs}\int_0^P f(x)e^{-sx}dx && \text{Because } f \text{ is } P\text{-periodic and } e^Ae^B=e^{A+B}.\\
&= \int_0^P f(x)e^{-sx}dx \sum_{n=0}^\infty r^n && \text{Common factor in summation. Define } r=e^{-Ps}.\\
&= \int_0^P f(x)e^{-sx}dx \frac{1}{1-r} && \text{Sum the geometric series.}\\
&= \frac{\int_0^P f(x)e^{-sx}dx}{1-e^{-Ps}} && \text{Substitute } r=e^{-Ps}.\\
&= \text{RHS} && \text{Periodic function identity verified.}
\end{aligned}$$

Left unmentioned here is the convergence of the infinite series on line 3 of the proof, which follows from f of exponential order.

Proof of Theorem 11 (convolution rule): The details use Fubini's integration interchange theorem for a planar unbounded region, and therefore this proof involves advanced calculus methods that may be outside the background of the reader. Modern calculus texts contain a less general version of Fubini's theorem for finite regions, usually referenced as *iterated integrals*. The unbounded planar region is written in two ways:

$$D = \{(r,t) : t \le r < \infty,\ 0 \le t < \infty\},$$
$$\mathbf{D} = \{(r,t) : 0 \le r < \infty,\ 0 \le r \le t\}.$$

Readers should pause here and verify that $D = \mathbf{D}$.

The change of variable $r = x + t$, $dr = dx$ is applied for fixed $t \geq 0$ to obtain the identity

$$e^{-st}\int_0^\infty g(x)e^{-sx}dx = \int_0^\infty g(x)e^{-sx-st}dx = \int_t^\infty g(r-t)e^{-rs}dr. \tag{2}$$

The left side of the convolution identity is expanded as follows:

$$\begin{aligned}
\text{LHS} &= \mathcal{L}(f(t))\mathcal{L}(g(t)) & \\
&= \int_0^\infty f(t)e^{-st}dt\int_0^\infty g(x)e^{-sx}dx & \text{Direct transform.}\\
&= \int_0^\infty f(t)\int_t^\infty g(r-t)e^{-rs}drdt & \text{Apply identity (2).}\\
&= \iint_D f(t)g(r-t)e^{-rs}drdt & \text{Fubini's theorem applied.}\\
&= \iint_{\mathcal{D}} f(t)g(r-t)e^{-rs}drdt & \text{Descriptions } D \text{ and } \mathcal{D} \text{ are the same.}\\
&= \int_0^\infty\int_0^r f(t)g(r-t)dt\,e^{-rs}dr & \text{Fubini's theorem applied.}
\end{aligned}$$

Then

$$\begin{aligned}
\text{RHS} &= \mathcal{L}\left(\int_0^t f(u)g(t-u)du\right) & \\
&= \int_0^\infty\int_0^t f(u)g(t-u)du\,e^{-st}dt & \text{Direct transform.}\\
&= \int_0^\infty\int_0^r f(u)g(r-u)du\,e^{-sr}dr & \text{Change variable names } r \leftrightarrow t.\\
&= \int_0^\infty\int_0^r f(t)g(r-t)dt\,e^{-sr}dr & \text{Change variable names } u \leftrightarrow t.\\
&= \text{LHS} & \text{Convolution identity verified.}
\end{aligned}$$

More on the Laplace Transform

Model conversion and engineering. A *differential equation model* for a physical system can be subjected to the Laplace transform in order to produce an *algebraic model* in the transform variable s. Lerch's theorem says that both models are equivalent, that is, the solution of one model gives the solution to the other model.

In electrical and computer engineering it is commonplace to deal *only* with the Laplace algebraic model. Engineers are in fact capable of having hour-long modeling conversations, during which differential equations are *never referenced*! Terminology for such modeling is necessarily specialized, which gives rise to new contextual meanings to the terms *input* and *output*. For example, an RLC-circuit would be discussed with *input*

$$F(s) = \frac{\omega}{s^2 + \omega^2},$$

and the listener must know that this expression is the Laplace transform of the t-expression $\sin \omega t$. Hence the RLC-circuit is driven by a sinusoindal input of natural frequency ω. During the modeling discourse, it could be that the *output* is

$$X(s) = \frac{1}{s+1} + \frac{10\omega}{s^2 + \omega^2}.$$

Lerch's equivalence says that $X(s)$ is the Laplace transform of $e^{-t} + 10 \sin \omega t$, but that is extra work, if all that is needed from the model is a statement about the transient and steady-state responses to the input.

z transforms

In the study of discrete-time signal and systems, we have thus far considered the time-domain and the frequency domain. The *z*-domain gives us a third representation. All three domains are related to each other.

A special feature of the z-transform is that for the signals and system of interest to us, all of the analysis will be in terms of ratios of polynomials. Working with these polynomials is relatively straight forward.

Definition of the *z*-Transform

- Given a finite length signal $x[n]$, the *z*-transform is defined as

$$\boxed{X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^{k}} \tag{7.1}$$

  where the sequence support interval is [0, *N*], and *z* is any complex number

- This transformation produces a new representation of $x[n]$ denoted $X(z)$

- Returning to the original sequence (*inverse z-transform*) $x[n]$ requires finding the coefficient associated with the *n*th power of z^{-1}

- Formally transforming from the time/sequence/n-domain to the z-domain is represented as

$$n\text{-Domain} \overset{z}{\leftrightarrow} \quad \text{omain}$$

$$x[n] = \sum_{k=0}^{N} x[k]\delta[n-k] \overset{z}{\leftrightarrow} X(z) = \sum_{k=0}^{N} x[k]z^{-k}$$

- A sequence and its z-transform are said to form a *z-transform pair* and are denoted

$$x[n] \overset{z}{\leftrightarrow} X(z) \tag{7.2}$$

  – In the sequence or n-domain the independent variable is n

  – In the z-domain the independent variable is z

<u>Example</u>: $x[n] = \delta[n-n_0]$

- Using the definition

$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} \delta[k-n_0]z^{-k} = z^{-n_0}$$

- Thus,

$$\delta[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}$$

Example: $x[n] = 2\delta[n] + 3\delta[n-1] + 5\delta[n-2] + 2\delta[n-3]$

- By inspection we find that

$$X(z) = 2 + 3z^{-1} + 5z^{-2} + 2z^{-3}$$

Example: $X(z) = 4 - 5z^{-2} + z^{-3} - 2z^{-4}$

- By inspection we find that

$$x[n] = 4\delta[n] - 5\delta[n-2] + \delta[n-3] - 2\delta[n-4]$$

- What can we do with the z-transform that is useful?

The z-Transform and Linear Systems

- The z-transform is particularly useful in the analysis and design of LTI systems

The z-Transform of an FIR Filter

- We know that for any LTI system with input $x[n]$ and impulse response $h[n]$, the output is

$$y[n] = x[n] * h[n] \tag{7.3}$$

- We are interested in the z-transform of $h[n]$, where for an FIR filter

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{7.4}$$

- To motivate this, consider the input

$$x[n] = z^n, -\infty < n < \infty \tag{7.5}$$

- The output $y[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k z^{n-k} \\ &= \sum_{k=0}^{M} b_k z^{n} z^{-k} = \left(\sum_{k=0}^{M} b_k z^{-k}\right) z^n \end{aligned} \tag{7.6}$$

- The term in parenthesis is the z-transform of $h[n]$, also known as the *system function* of the FIR filter
- Like $H(e^{j\omega})$ was defined in Chapter 6, we define the system function as

$$\boxed{H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}} \tag{7.7}$$

- The z-transform pair we have just established is

$$h[n] \overset{z}{\leftrightarrow} H(z)$$

$$\sum_{k=0}^{M} b_k \delta[n-k] \overset{z}{\leftrightarrow} \sum_{k=0}^{M} b_k z^{-k}$$

- Another result, similar to the frequency response result, is

$$y[n] = h[n] * z^n = H(z) z^n \tag{7.8}$$

– Note if $z = e^{j\hat{\omega}}$, we in fact have the frequency response result of Chapter 6

- The system function is an Mth degree polynomial in complex variable z
- As with any polynomial, it will have M roots or *zeros*, that is there are M values z_0 such that $H(z_0) = 0$

– These M zeros completely define the polynomial to within a gain constant (scale factor), i.e.,

$$\begin{aligned} H(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\ &= (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \cdots (1 - z_M z^{-1}) \\ &= \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{z^M} \end{aligned}$$

where z_k, $k = 1, \ldots, M$ denote the zeros

<u>Example</u>: Find the Zeros of

$$h[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$$

- The z-transform is

$$\begin{aligned} H(z) &= 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} \\ &= \left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right) \\ &= \left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right) / z^2 \end{aligned}$$

- The zeros of $H(z)$ are -1/2 and +1/3
- The difference equation

$$y[n] = 6x[n] + x[n-1] - x[n-2]$$

has the same zeros, but a different scale factor;

proof:

Properties of the *z*-Transform

- The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses

The Superposition (Linearity) Property

$$\boxed{ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z)} \tag{7.9}$$

proof

$$\begin{aligned} X(z) &= \sum_{n=0} (ax_1[n] + bx_2[n])z^{-1} \\ &= a\sum_{n=0}^{N} x_1[n]z^{-1} + b\sum_{n=0}^{N} x_2[n]z^{-1} \\ &= aX_1(z) + bX_2(z) \end{aligned}$$

The Time-Delay Property

$$x[n-1] \overset{z}{\leftrightarrow} z^{-1}X(z) \tag{7.10}$$

and

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z) \tag{7.11}$$

proof: Consider

$$X(z) = \alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_N z^{-N}$$

then

$$\begin{aligned} x[n] &= \sum_{k=0}^{N} \alpha_k \delta[n-k] \\ &= \alpha_0\delta[n] + \alpha_1\delta[n-1] + \cdots + \alpha_N\delta[n-N] \end{aligned}$$

Let

$$\begin{aligned} Y(z) &= z^{-1}X(z) \\ &= \alpha_0 z^{-1} + \alpha_1 z^{-2} + \cdots + \alpha_N z^{-N-1} \end{aligned}$$

so

$$\begin{aligned} y[n] &= \alpha_0\delta[n-1] + \alpha_1\delta[n-2] + \cdots + \alpha_N\delta[n-N-1] \\ &= x[n-1] \end{aligned}$$

Similarly

$$\begin{aligned} Y(z) &= z^{-n_0}X(z) \\ \Rightarrow y[n] &= x[n-n_0] \end{aligned}$$

A General z-Transform Formula

- We have seen that for a sequence $x[n]$ having support interval $0 \le n \le N$ the z-transform is

$$X(z) = \sum_{n=0}^{N} x[n]z^{-n} \tag{7.12}$$

- This definition extends for doubly infinite sequences having support interval $-\infty \le n \le \infty$ to

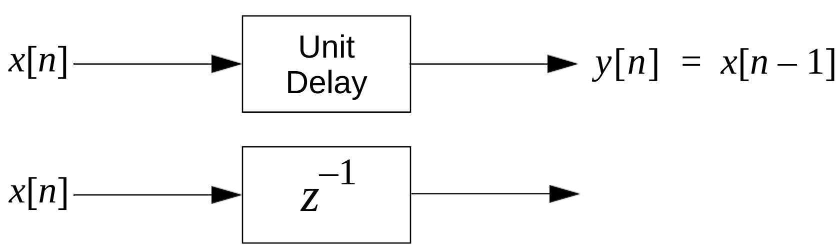
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{7.13}$$

  – There will be discussion of this case in Chapter 8 when we deal with infinite impulse response (IIR) filters

The z-Transform as an Operator

The z-transform can be considered as an operator.

Unit-Delay Operator



- In the case of the unit delay, we observe that

$$y[n] = \underbrace{z^{-1}}_{\text{unit delay operator}}\{x[n]\} = x[n-1] \quad (7.14)$$

which is motivated by the fact that $Y(z) = z^{-1}X(z)$

- Similarly, the filter

$$y[n] = x[n] - x[n-1]$$

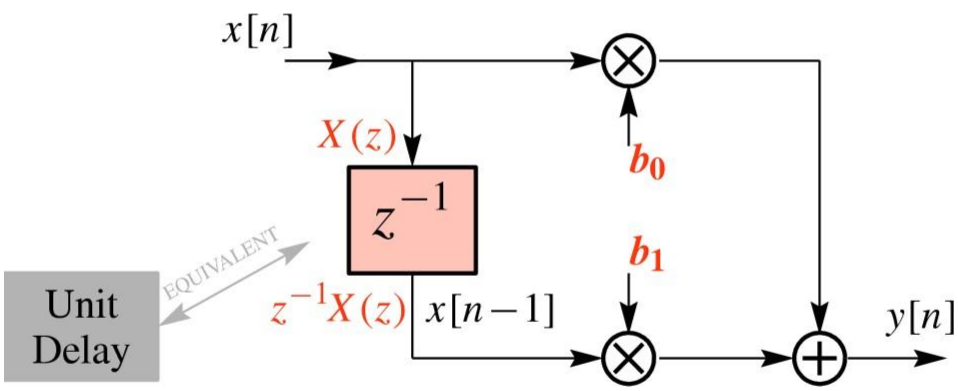
can be viewed as the operator

$$y[n] = (1 - z^{-1})\{x[n]\} = x[n] - x[n-1]$$

since

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

Example: Two-Tap FIR



- Using the operator convention, we can write by inspection that

$$Y(z) = b_0X(z) + b_1z^{-1}X(z)$$
$$y[n] = b_0x[n] + b_1x[n-1]$$

Convolution and the z-Transform

- The impulse response of the unity delay system is

$$h[n] = \delta[n-1]$$

and the system output written in terms of a convolution is

$$y[n] = x[n]*\delta[n-1] = x[n-1]$$

- The system function (z-transform of $h[n]$) is

$$H(z) = z^{-1}$$

and by the previous unit delay analysis,

$$Y(z) = z^{-1}X(z)$$

- We observe that

$$\boxed{Y(z) = H(z)X(z)} \tag{7.15}$$

<u>proof</u>:

$$y[n] = x[n]*h[n] = \sum_{k=0}^{M} h[k]x[n-k] \tag{7.16}$$

We now take the z-transform of both sides of (7.16) using superposition and the general delay property

$$\begin{aligned} Y(z) &= \sum_{k=0}^{M} h[k](z^{-k}X(z)) \\ &= \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z) \end{aligned} \tag{7.17}$$

- Note: For the case of $x[n]$ a finite duration sequence, $X(z)$ is a polynomial, and $H(z)X(z)$ is a product of polynomials in z^{-1}

Example: Convolving Finite Duration Sequences

- Suppose that

$$x[n] = 2\delta[n] - 3\delta[n-2] + 4\delta[n-3]$$
$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

- We wish to find $y[n]$ by first finding $Y(z)$
- We begin by z-transforming each of the sequences

$$X(z) = 2 - 3z^{-2} + 4z^{-3}$$
$$H(z) = 1 + 2z^{-1} + z^{-2}$$

- We find $Y(z)$ by direct multiplication

$$\begin{aligned} Y(z) &= (2 - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} + z^{-2}) \\ &= 2 + 4z^{-1} - z^{-2} - 2z^{-3} + 5z^{-4} + 4z^{-5} \end{aligned}$$

- We find $y[n]$ using the delay property on each of the terms of $Y(z)$

$$\begin{aligned} y[n] &= 2\delta[n] + 4\delta[n-1] - \delta[n-2] \\ &\quad - 2\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] \end{aligned}$$

Convolve directly?

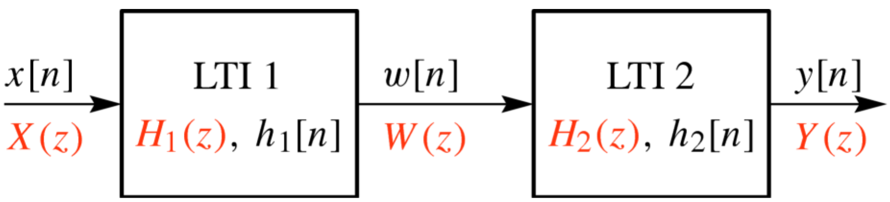
- This section has established the very important result that polynomial multiplication can be used to replace sequence convolution, when we work in the *z*-domain, i.e.,

z-Transform Convolution Theorem

$$y[n] = h[n]*x[n] \overset{z}{\leftrightarrow} H(z)X(z) \;=\; Y(z)$$

Cascading Systems

- We have seen cascading of systems in the time-domain and the frequency domain, we now consider the *z*-domain



- We know from the convolution theorem that

$$W(z) \;=\; H_1(z)X(z)$$

- It also follows that

$$Y(z) \;=\; H_2(z)W(z)$$

so by substitution

$$\begin{aligned} Y(z) &= [H_2(z)H_1(z)]X(z) \\ &= [H_1(z)H_2(z)]X(z) \end{aligned} \tag{7.18}$$

- In summary, when we cascade two LTI systems, we arrive at the cascade impulse response as a cascade of impulse responses in the time-domain and a product of the z-transforms in the *z*-domain

$$\boxed{h[n] = h_1[n]*h_2[n] \overset{z}{\leftrightarrow} H_1(z)H_2(z) = H(z)}$$

Factoring *z*-Polynomials

- Multiplying *z*-transforms creates a cascade system, so factoring must create subsystems

---

Example: $H(z) = 1 + 3z^{-1} - 2z^{-2} + z^{-3}$

- Since $H(z)$ is a third-order polynomial, we should be able to factor it into a first degree and second degree polynomial
- We can use the MATLAB function `roots()` to assist us

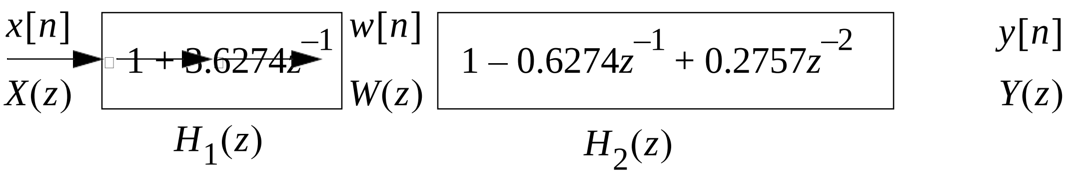
```
>> p = roots([1 3 -2 1])

p = -3.6274
     0.3137 + 0.4211i
     0.3137 - 0.4211i

>> conv([1 -p(2)],[1 -p(3)])

ans =  1.0000    -0.6274      0.2757 - 0.0000i
```

- With one real root, the logical factoring is to create two polynomials as follows

$$H_1(z) = 1 + 3.6274z^{-1}$$

$$H_2(z) = (1 - (0.3137 + j0.4211)z^{-1})$$
$$(1 - (0.3137 - j0.4211)z^{-1})$$
$$= 1 - 0.6274z^{-1} + 0.2757z^{-2}$$

- The cascade system is thus:



- As a check we can multiply the polynomials

```
>> conv([1 -p(1)],conv([1 -p(2)],[1 -p(3)]))

ans = 1.0000, 3.0000, -2.0000-0.0000i, 1.0000-0.0000i
```

- The difference equations for each subsystem are

$$w[n] = x[n] + 3.6274x[n-1]$$
$$y[n] = w[n] - 0.6274w[n-1] + 0.2757w[n-2]$$

Deconvolution/Inverse Filtering

- In a two subsystems cascade can the second system undo the action of the first subsystem?
- For the output to equal the input we need $H(z) = 1$
- We thus desire

$$H_1(z)H_2(z) = 1 \text{ or } H_2(z) = \frac{1}{H_1(z)}$$

<u>Example</u>: $H_1(z) = 1 - az^{-1}$, $|a| < 1$

- The inverse filter is

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - az^{-1}}$$

- This is no longer an FIR filter, it is an infinite impulse response (IIR) filter, which is the topic of Chapter 8
- We can approximate $H_2(z)$ as an FIR filter via long division

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad a^3z^{-3}
\end{array}$$

- An $M + 1$ term approximation is

$$H_2(z) = \sum_{k=0}^{M} a^k z^{-k}$$

  – Recall the deconvolution filter of Lab 8?

Relationship Between the z-Domain and the Frequency Domain

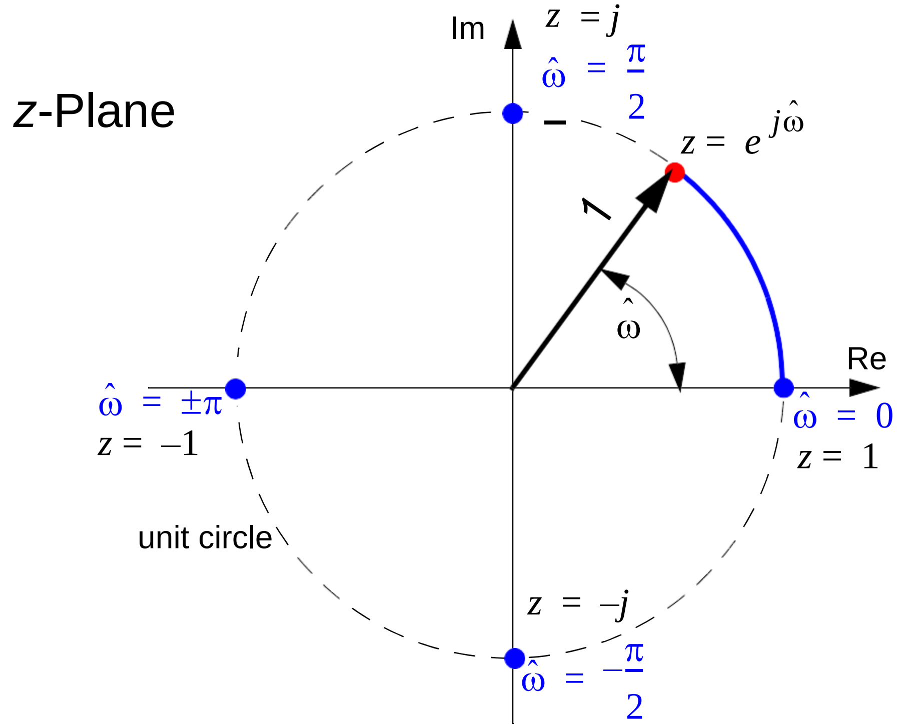
$\hat{\omega}$ - Domain		z - Domain
$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$	versus	$H(z) = \sum_{k=0}^{M} b_k z^{-k}$

- Comparing the above we see that the connection is setting $z = e^{j\hat{\omega}}$ in $H(z)$, i.e.,

$$\boxed{H(e^{j\hat{\omega}}) = H(z)\big|_{z = e^{j\hat{\omega}}}} \tag{7.19}$$

The z-Plane and the Unit Circle

- If we consider the z-plane, we see that $H(e^{j\hat{\omega}})$ corresponds to evaluating $H(z)$ on the unit circle



- From this interpretation we also can see why $H(e^{j\hat{\omega}})$ is periodic with period 2π
  - As $\hat{\omega}$ increases it continues to sweep around the unit circle over and over again

The Zeros and Poles of $H(z)$

- Consider

$$H(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} \tag{7.20}$$

where we have assumed that $b_0 = 1$

- Factoring $H(z)$ results in

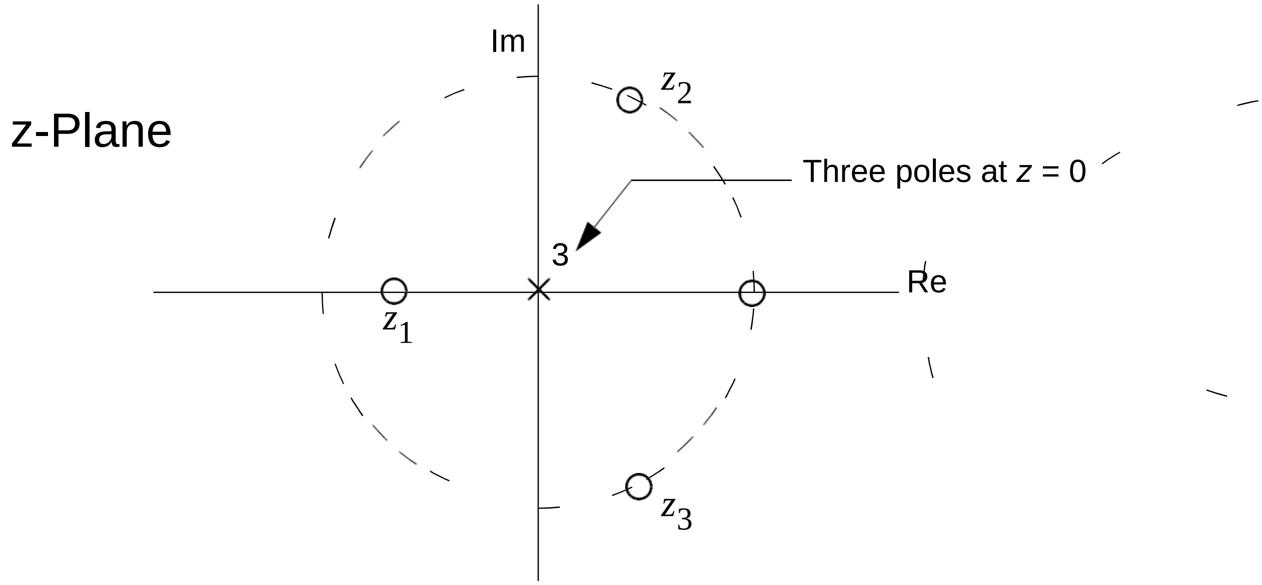
$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1}) \tag{7.21}$$

- Multiplying by z^3/z^3 allows to write $H(z)$ in terms of positive powers of z

$$\begin{aligned} H(z) &= \frac{z^3 + b_1 z^2 + b_2 z^1 + b_3 z^0}{z^3} \\ &= \frac{(z - z_1)(z - z_2)(z - z_3)}{z^3} \end{aligned} \tag{7.22}$$

- The *zeros* are the locations where $H(z) = 0$, i.e., z_1, z_2, z_3
- The *poles* are where $H(z) \to \infty$, i.e., $z \to 0$
- <u>Note</u> that the poles and zeros only determine $H(z)$ to within a constant; recall the example on page 7-5

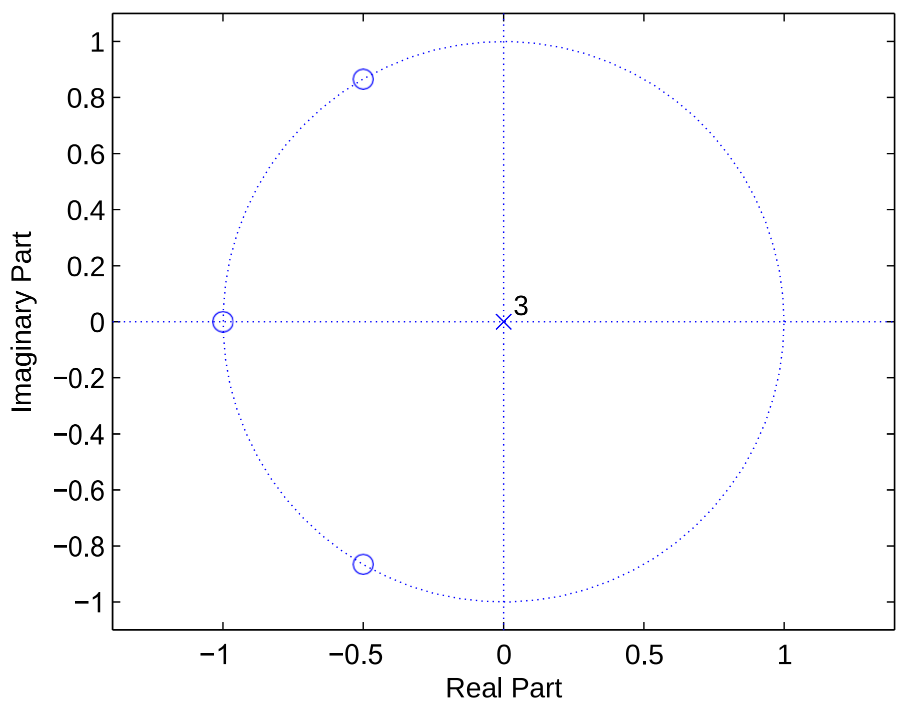
- A *pole-zero plot* displays the pole and zero locations in the *z*-plane



Example: $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

- MATLAB has a function that supports the creation of a pole-zero plot given the system function coefficients

```
>> zplane([1 2 2 1],1)
```



The Significance of the Zeros of *H*(*z*)

- The difference equation is the actual time domain means for calculating the filter output for a given filter input
- The difference equation coefficients are the polynomial coefficients in $H(z)$
- For $x[n] = z_0^n$ we know that

$$y[n] = H(z_0)z_0^n, \tag{7.23}$$

so in particular if z_0 is one of the zeros of $H(z)$, $H(z_0) = 0$ and the output $y[n] = 0$

- <u>If a zero lies on the unit circle</u> then the output will be zero for a sinusoidal input of the form

$$x[n] = z_0^n = (e^{j\hat{\omega}_0})^n = e^{j\hat{\omega}_0 n} \tag{7.24}$$

where $\hat{\omega}_0$ is the angle of the zero relative to the real axis, which is also the frequency of the corresponding complex sinusoid; why?

$$y[n] = \left[H(z)\Big|_{z = e^{j\hat{\omega}_0}} \right] e^{j\hat{\omega}_0 n} = 0 \tag{7.25}$$

Nulling Filters

- The special case of zeros on the unit circle allows a filter to *null*/block/*annihilate* complex sinusoids that enter the filter at frequencies corresponding to the angles the zeros make with respect to the real axis in the z-plane

- The nulling property extends to real sinusoids since they are composed of two complex sinusoids at $\pm\hat{\omega}_0$, and zeros not on the real axis will always occur in conjugate pairs if the filter coefficients are real
- This nulling/annihilating property is useful in rejecting unwanted jamming and interference signals in communications and radar applications

---

<u>Example</u>: $H(z) = 1 - 2\cos(\hat{\omega}_0)z^{-1} + z^{-2}$, $x[n] = \cos(\hat{\omega}_0 n)$

- Factoring $H(z)$ we find that

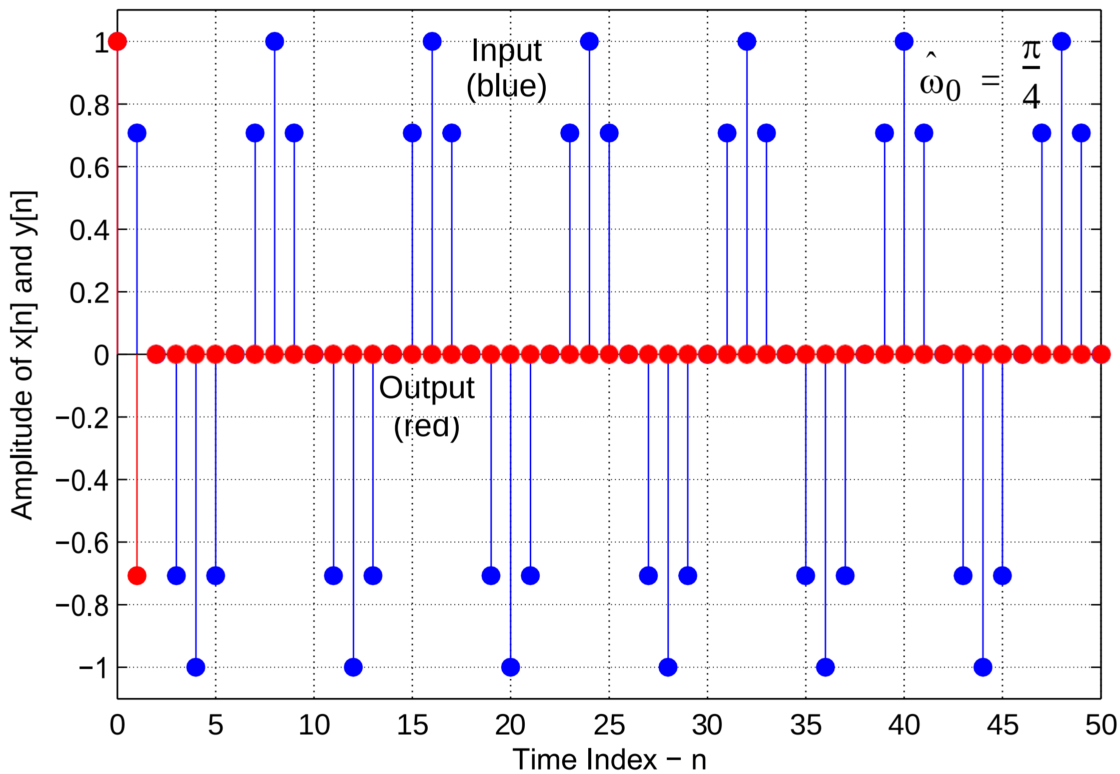
$$H(z) = \Big(1 - \underbrace{e^{j\hat{\omega}_0}}_{z_1} z^{-1}\Big)\Big(1 - \underbrace{e^{-j\hat{\omega}_0}}_{z_2} z^{-1}\Big)$$

- Expanding $x[n]$ we see that

$$x[n] = \frac{1}{2}e^{-j\hat{\omega}_0 n} + \frac{1}{2}e^{j\hat{\omega}_0 n}$$

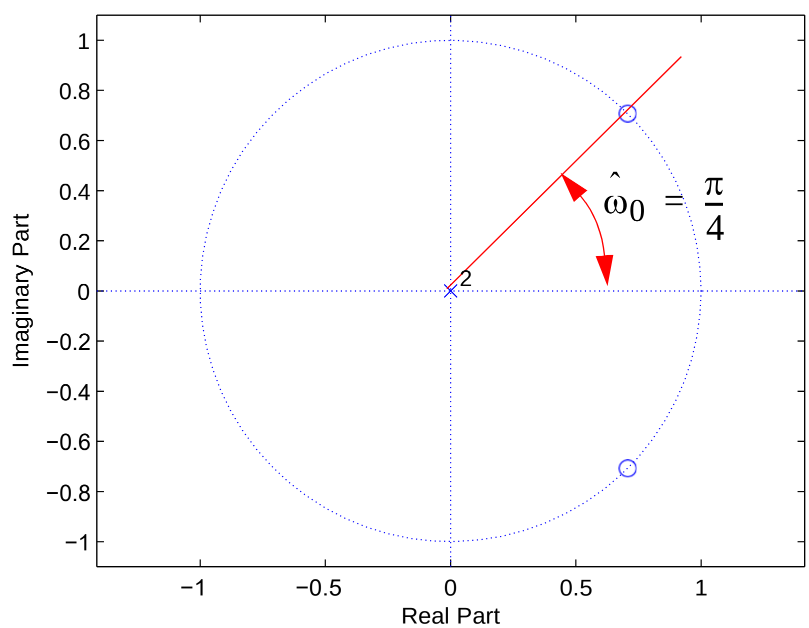
- The nulling action of $H(z)$ at $\pm\hat{\omega}_0$ will remove the signal from the filter output
- We can set up a simple simulation in MATLAB to verify this

```
>> n = 0:100;
>> w0 = pi/4;
>> x = cos(w0*n);
>> y = filter([1 -2*cos(w0) 1],1,x);
>> stem(n,x,'filled')
>> hold
Current plot held
>> stem(n,y,'filled','r')
>> axis([0 50 -1.1 1.1]); grid
```



- Since the input is applied at $n = 0$, we see a small transient while the filter settles to the final output, which in this case is zero

```
>> zplane([1 -2*cos(w0) 1],1)% check the pole-zero plot
```



Graphical Relation Between z and $\hat{\omega}$

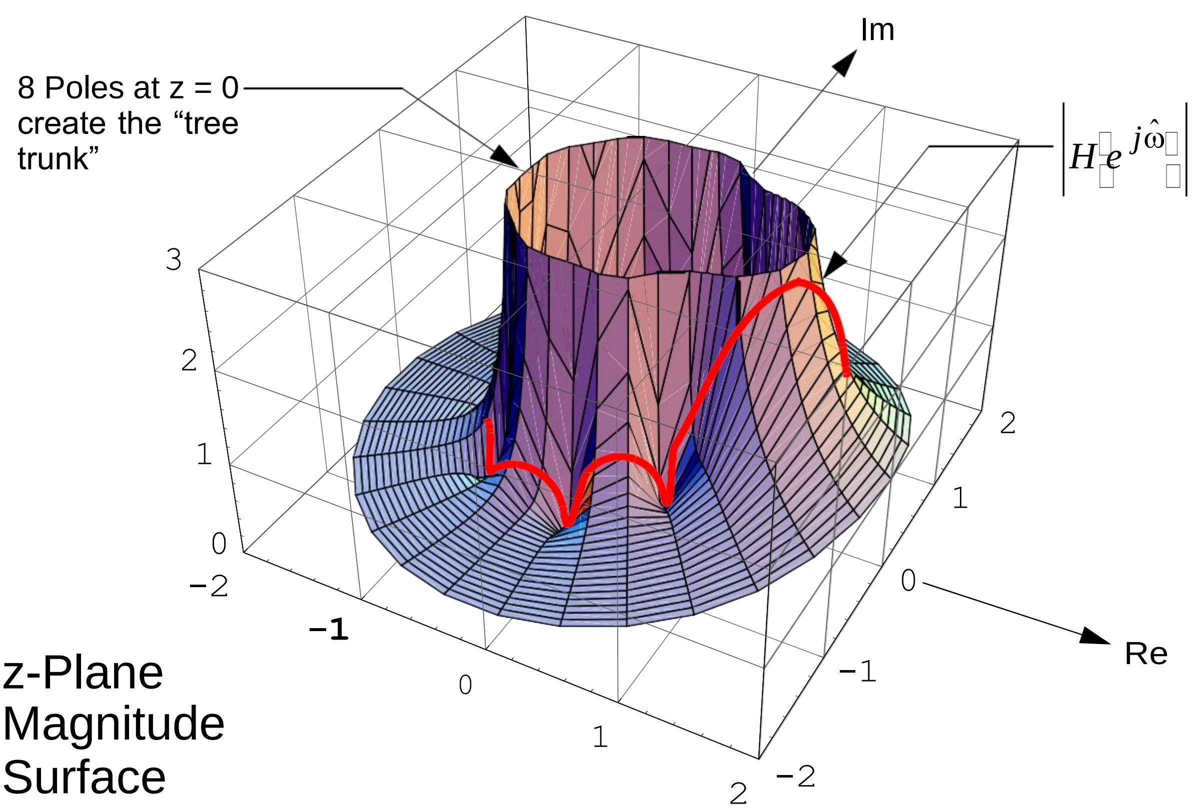
- When we make the substitution $z = e^{j\hat{\omega}}$ in $H(z)$ we know that we are evaluating the z-transform on the unit circle and thus obtain the frequency response
- If we plot say $|H(z)|$ over the entire z-plane we can visualize how cutting out the response on just the unit circle, gives us the frequency response magnitude

---

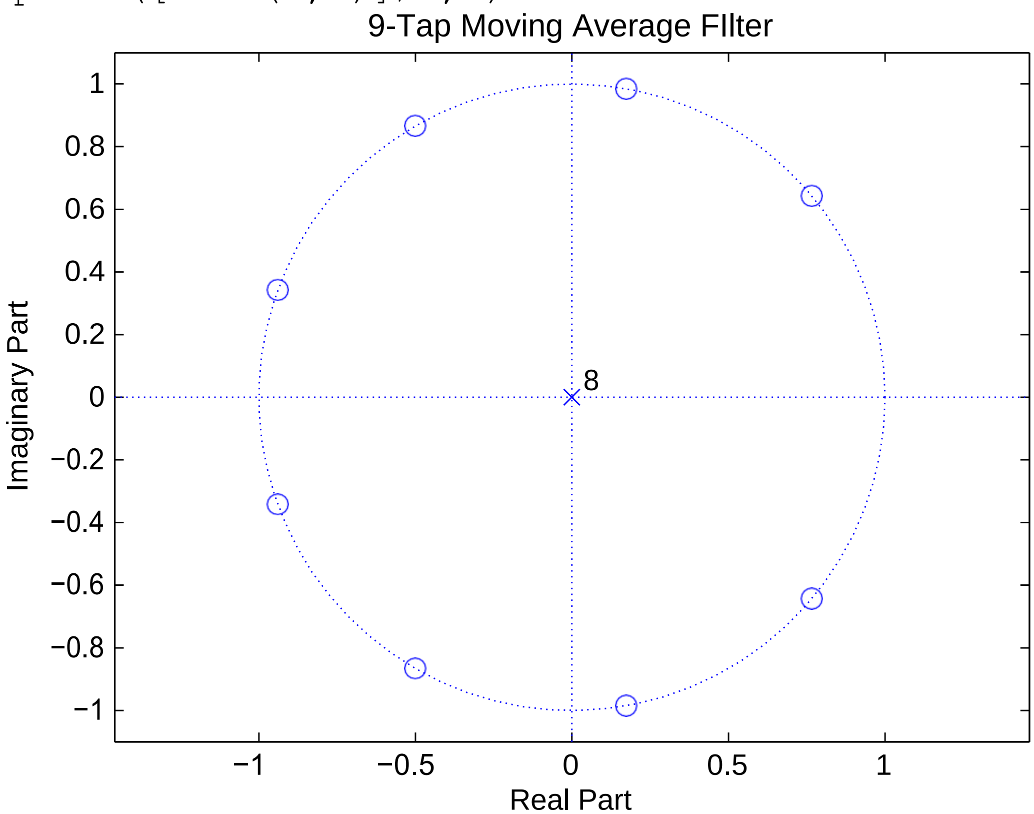
<u>Example</u>: $L = 9$ Moving Average Filter (9 taps/8th-order)

- Here we have

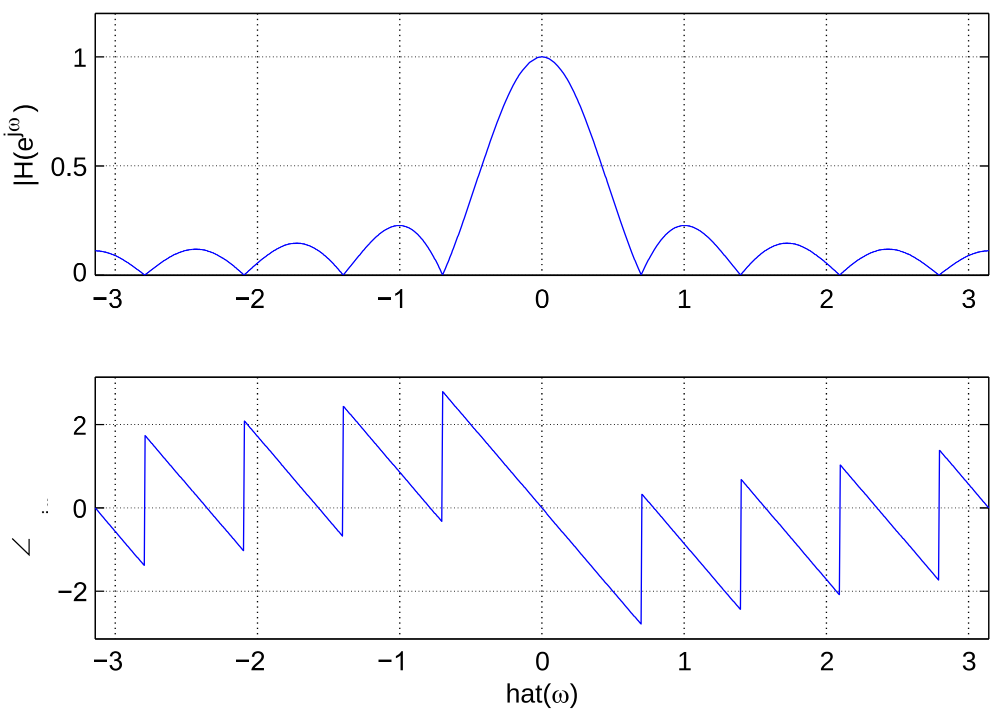
$$H(z) = \frac{1}{9}\sum_{k=0}^{9-1} z^{-k} = \frac{1}{9}\prod_{k=1}^{8}(1 - e^{-j2\pi k/9}z^{-1})$$



```
>> zplane([ones(1,9)]/9,1)
```




```
>> w = -pi:(pi/500):pi;
>> H = freqz([ones(1,9)/9],1,w);
```



Useful Filters

The *L*-Point Moving Average Filter

- The L-point moving average (running sum) filter has

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \tag{7.26}$$

and system function (z-transform of the impulse response)

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} \tag{7.27}$$

- The sum in (7.27) can be simplified using the geometric series sum formula

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} = \frac{1}{L}\cdot\frac{1-z^{-L}}{1-z^{-1}} = \frac{1}{L}\cdot\frac{z^{L}-1}{z^{L-1}(z-1)} \tag{7.28}$$

- Notice that the zeros of $H(z)$ are determined by the roots of the equation

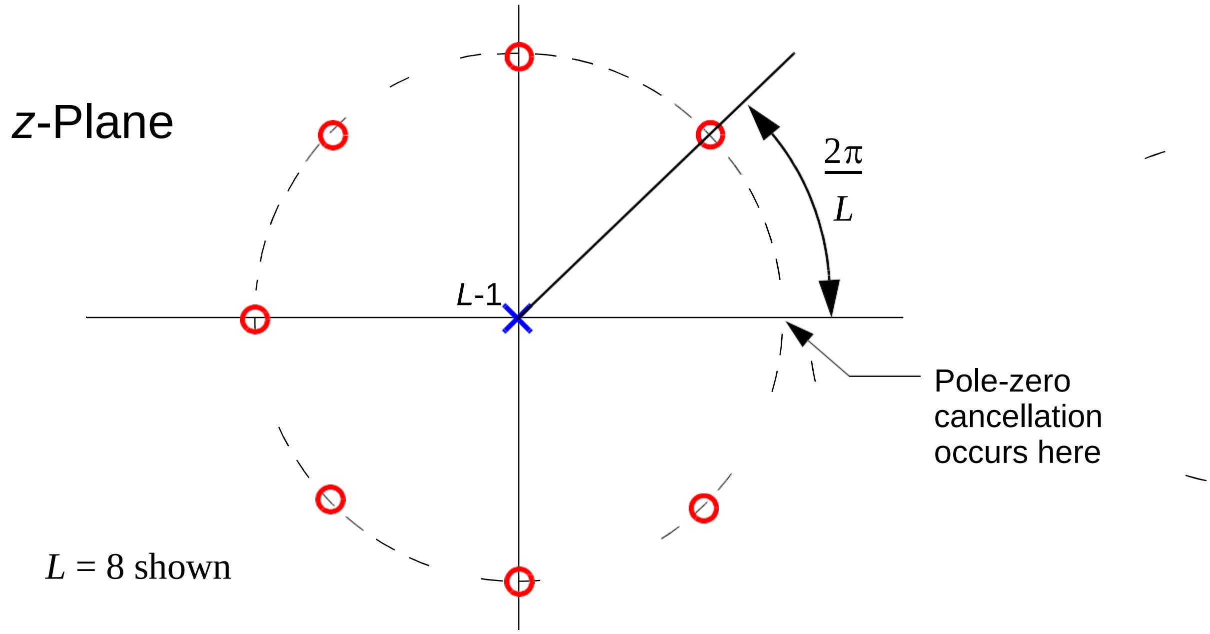
$$z^{L} - 1 = 0 \Rightarrow z^{L} = 1 \tag{7.29}$$

- The roots of this equation can be found by noting that $e^{j2\pi k} = 1$ for k any integer, thus the roots of (7.29) (zeros of (7.28)) are

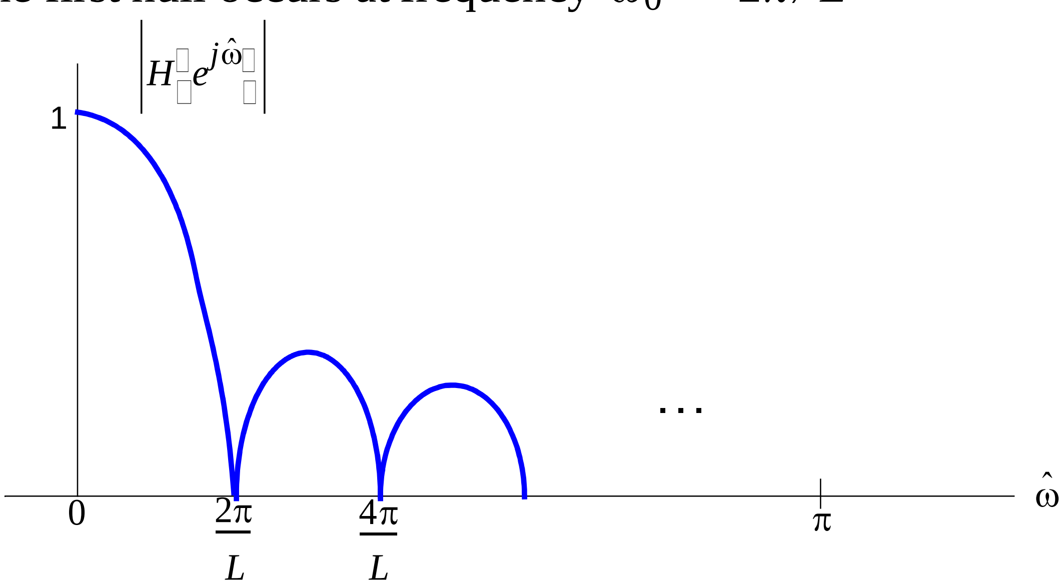
$$z_k = e^{j2\pi k/L},\ k = 0, 1, 2, \ldots, L-1 \tag{7.30}$$

- These roots are referred to as the L roots of unity

- One of the zeros sits at $z = 1$, but there is also a pole at $z = 1$, so there is a pole-zero cancellation, meaning that the pole-zero plot of $H(z)$ corresponds to the L-roots of unity, less the root at $z = 0$



- We have seen the frequency response of this filter before
- The first null occurs at frequency $\hat{\omega}_0 = 2\pi/L$



A Complex Bandpass Filter

see text

A Bandpass Filter with Real Coefficients

see text

Practical Filter Design

- Here we will use `fdatool` from the MATLAB signal processing toolbox to design an FIR filter

Properties of Linear-Phase Filters

- A class of FIR filters having symmetrical coefficients, i.e., $b_k = b_{M-k}$ for $k = 0, 1, \ldots, M$ has the property of linear phase

The Linear Phase Condition

- For a filter with symmetrical coefficients we can show that $H(e^{j\hat{\omega}})$ is of the form

$$H(e^{j\hat{\omega}}) = R(e^{j\hat{\omega}})e^{-j\omega M/2} \tag{7.31}$$

  where $R(e^{j\hat{\omega}})$ is a real function

- The fact that $R(e^{j\hat{\omega}})$ is real means that the phase of $H(e^{j\hat{\omega}})$ is a linear function of frequency plus the possibility of $\pm\pi$ phase jumps whenever $R(e^{j\hat{\omega}})$ passes through zero

---

<u>Example</u>: $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_1 z^{-3} + b_0 z^{-4}$

- By factoring out z^{-2} we can write

$$H(z) = [b_0(z^2 + z^{-2}) + b_1(z^1 + z^{-1}) + b_2]z^{-2}$$

- We now move to the frequency response by letting $z \to e^{j\hat{\omega}}$

$$H(e^{j\hat{\omega}}) = [2b_0\cos(2\hat{\omega}) + 2b_1\cos(\hat{\omega}) + b_2]e^{-j\hat{\omega}4/2}$$

- Note that here we have $M = 4$, so we see that the linear phase term is indeed of the form $e^{-j\hat{\omega}M/2}$ and the real function $R(e^{j\hat{\omega}})$ is of the form

$$R(e^{j\hat{\omega}}) = b_2 + 2[b_0\cos(2\hat{\omega}) + b_1\cos(\hat{\omega})]$$

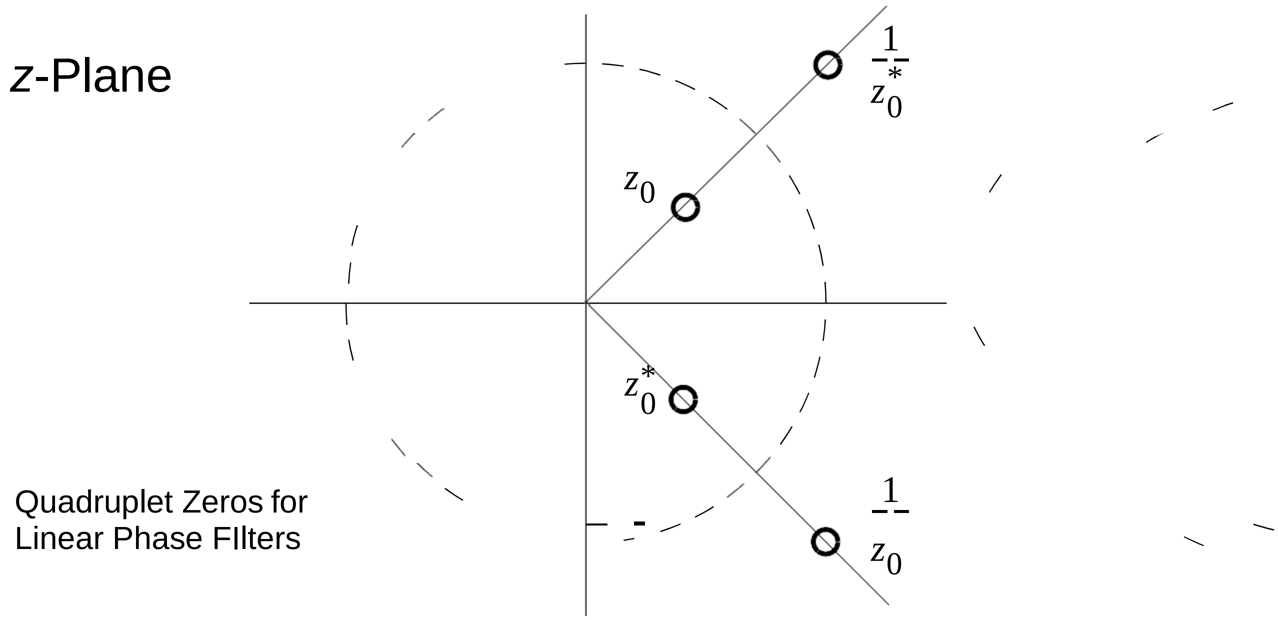
Locations of the Zeros of FIR Linear-Phase Systems

- Further study of $H(z)$ for the case of symmetric coefficients reveals that

$$H(1/z) = z^M H(z) \tag{7.32}$$

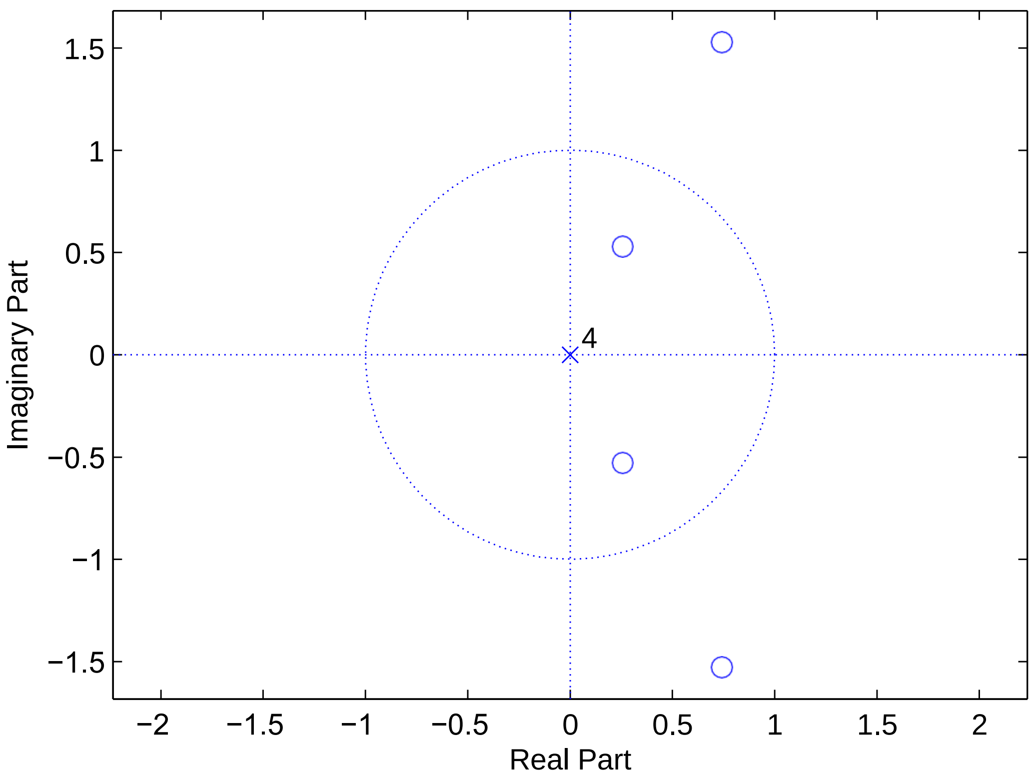
- A consequence of this condition is that for $H(z)$ having a zero at z_0 it will also have a zero at $1/z_0$
- Assuming the filter has real coefficients, complex zeros occur in conjugate pairs, so the even symmetry condition further implies that the zeros occur as quadruplets

$$\left\{ z_0, z_0^*, \frac{1}{z_0}, \frac{1}{z_0^*} \right\}$$



Example: $H(z) = 1 - 2z^{-1} + 4z^{-2} - 2z^{-3} + z^{-4}$

```
>> zplane([1 -2 4 -2 1],1)
```



UNIT-4

Sampling and Reconstruction

Digital hardware, including computers, take actions in discrete steps. So they can deal with discrete-time signals, but they cannot directly handle the continuous-time signals that are prevalent in the physical world. This chapter is about the interface between these two worlds, one continuous, the other discrete. A discrete-time signal is constructed by **sampling** a continuous-time signal, and a continuous-time signal is **reconstructed** by **interpolating** a discrete-time signal.

Sampling

A sampler for complex-valued signals is a system

$$Sampler_T : [Reals \to Complex] \to [Integers \to Complex], \tag{11.1}$$

where T is the **sampling interval** (it has units of seconds/sample). The system is depicted in figure 11.1. The **sampling frequency** or **sample rate** is $f_s = 1/T$, in units of samples/second (or sometimes, Hertz), or $\omega_s = 2\pi/T$, in units radians/second. If $y = Sampler_T(x)$ then y is defined by

$$\forall\, n \in Integers, \quad y(n) = x(nT). \tag{11.2}$$

Sampling a sinusoid

Let x: $Reals \to Reals$ be the sinusoidal signal

$$\forall\, t \in Reals, \quad x(t) = \cos(2\pi f t), \tag{11.3}$$

Figure 11.1: Sampler.

Basics: Units

Recall that frequency can be given with any of various units. The units of the f in (11.3) and (11.4) are Hertz, or cycles/second. In (11.3), it is sensible to give the frequency as $\omega = 2\pi f$, which has units of radians/second. The constant 2π has units of radians/cycle, so the units work out. Moreover, the time argument t has units of seconds, so the argument to the cosine function, $2\pi ft$, has units of radians, as expected.

In the discrete time case (11.4), it is sensible to give the frequency as $2\pi fT$, which has units of radians/sample. The sampling interval T has units of seconds/sample, so again the units work out. Moreover, the integer n has units of samples, so again the argument to the cosine function, $2\pi fnT$, has units of radians, as expected.

In general, when discussing continuous-time signals and their sampled discrete-time signals, it is important to be careful and consistent in the units used, or considerable confusion can result. Many texts talk about **normalized frequency** when discussing discrete-time signals, by which they simply mean frequency in units of radians/sample. This is **normalized** in the sense that it does not depend on the sampling interval.

where f is the frequency of the sinewave in Hertz. Let $y = Sampler_T(x)$. Then

$$\forall\, n \in Integers, \quad y(n) = \cos(2\pi fnT). \tag{11.4}$$

Although this looks similar to the continuous-time sinusoid, there is a fundamental difference. Because the index n is discrete, it turns out that the frequency f is indistinguishable from frequency $f + f_s$ when looking at the discrete-time signal. This phenomenon is called **aliasing**.

Aliasing

Consider another sinusoidal signal u given by

$$\forall\, t \in Reals, \quad u(t) = \cos(2\pi(f + Nf_s)t),$$

where N is some integer and $f_s = 1/T$. If $N = 0$, then this signal is clearly different from x in (11.3). Let

$$w = Sampler_T(u).$$

Then for all $n \in Integers$,

$$w(n) = \cos(2\pi(f + Nf_s)nT) = \cos(2\pi fnT + 2\pi Nn) = \cos(2\pi fnT) = y(n),$$

because Nn is an integer. Thus, even though $u = x$, $Sampler_T(u) = Sampler_T(x)$. Thus, after being sampled, the signals x and u are indistinguishable. This phenomenon is called **aliasing**, presumably because it implies that any discrete-time sinusoidal signal has many continuous-time identities (its "identity" is presumably its frequency).

Example 11.1: A typical sample rate for voice signals is $f_s = 8000$ samples/second, so the sampling interval is $T = 0.125$ msec/sample. A continuous-time sinusoid with frequency 440 Hz, when sampled at this rate, is indistinguishable from a continuous-time sinusoid with frequency 8,440 Hz, when sampled at this same rate.

Example 11.2: Compact discs are created by sampling audio signals at $f_s = 44,100$ Hz, so the sampling interval is about $T = 22.7$ μsec/sample. A continuous-time sinusoid with frequency 440 Hz, when sampled at this rate, is indistinguishable from a continuous-time sinusoid with frequency 44,540 Hz, when sampled at this same rate.

The frequency domain analysis of the previous chapters relied heavily on complex exponential signals. Recall that a cosine can be given as a sum of two complex exponentials, using Euler's relation,

$$\cos(2\pi f t) = 0.5(e^{i2\pi f t} + e^{-i2\pi f t}).$$

One of the complex exponentials is at frequency f, an the other is at frequency $-f$. Complex exponential exhibit the same aliasing behavior that we have illustrated for sinusoids.

Let x: *Reals* $\to$ *Complex* be

$$\forall\, t \in Reals, \quad x(t) = e^{i2\pi f t}$$

where f is the frequency in Hertz. Let $y = Sampler_T(x)$. Then for all n in *Integers*,

$$y(n) = e^{i2\pi f nT}$$

Consider another complex exponential signal u,

$$u(t) = e^{i2\pi(f+Nf_s)t}$$

where N is some integer. Let

$$w = Sampler_T(u).$$

Then for all $n \in Integers$,

$$w(n) = e^{i2\pi(f+Nf_s)nT} = e^{i2\pi f nT} e^{i2\pi N f_s nT} = e^{i2\pi f nT} = y(n),$$

because $e^{i2\pi N f_s nT} = 1$. Thus, as with sinusoids, when we sample a complex exponential signal with frequency f at sample rate f_s, it is indistinguishable from one at frequency $f + f_s$ (or $f + N f_s$ for any integer N).

There is considerably more to this story. Mathematically, aliasing relates to the periodicity of the frequency domain representation (the DTFT) of a discrete-time signal. We will also see that the effects of aliasing on real-valued signals (like the cosine, but unlike the complex exponential) depend strongly on the conjugate symmetry of the DTFT as well.

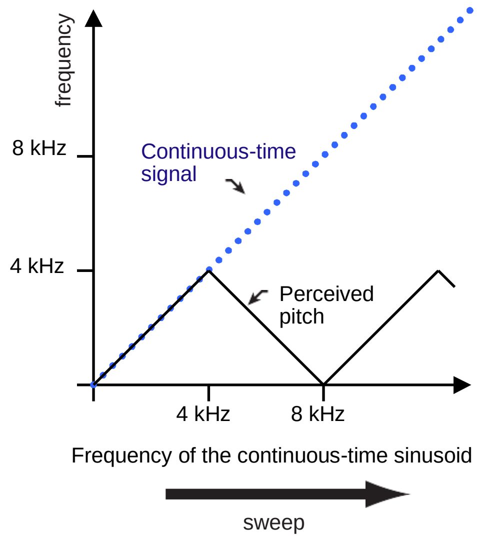


Figure 11.2: As the frequency of a continuous signal increases beyond the Nyquist frequency, the perceived pitch starts to drop.

Perceived pitch experiment

Consider the following experiment.[1] Generate a discrete-time audio signal with an 8,000 samples/second sample rate according to the formula (11.4). Let the frequency f begin at 0 Hz and sweep upwards through 4 kHz to (at least) 8 kHz. Use the audio output of a computer to listen to the resulting sound. The result is illustrated in figure 11.2. As the frequency of the continuous-time sinusoid rises, so does the perceived pitch, until the frequency reaches 4 kHz. At that point, the perceived pitch begins to fall rather than rise, even as the frequency of the continuous-time sinusoid continues to rise. It will fall until the frequency reaches 8 kHz, at which point no sound is heard at all (the perceived pitch is 0 Hz). Then the perceived pitch begins to rise again.

That the perceived pitch rises from 0 after the frequency f rises above 8000 Hz is not surprising. We have already determined that in a discrete-time signal, a frequency of f is indistinguishable from a frequency $f + 8000$, assuming the sample rate is 8,000 samples/second. But why does the perceived pitch drop when f rises above 4 kHz?

The frequency 4 kHz, $f_s/2$, is called the **Nyquist frequency**, after Harry Nyquist, an engineer at Bell Labs who, in the 1920s and 1930s, laid much of the groundwork for digital transmission of information. The Nyquist frequency turns out to be a key threshold in the relationship between discrete-time and continuous-time signals, more important even than the sampling frequency. Intuitively, this is because if we sample a sinusoid with a frequency below the Nyquist frequency (below half the sampling frequency), then we take at least two samples per cycle of the sinusoid. It should be intuitively appealing that taking at least two samples per cycle of a sinusoid has some key

[1]This experiment can be performed at http://www.eecs.berkeley.edu/ẽal/eecs20/week13/aliasing.html. Similar experiments are carried out in lab C.11.

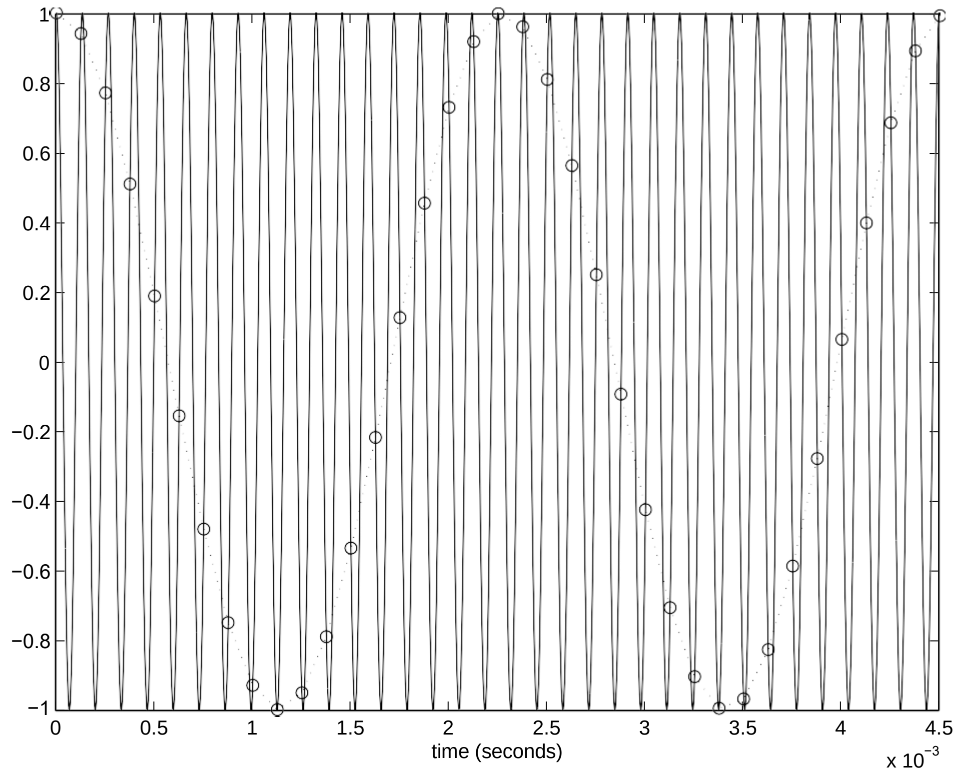


Figure 11.3: A sinusoid at 7.56 kHz and samples taken at 8 kHz.

significance. The two sample minimum allows the samples to capture the oscillatory nature of the sinusoid. Fewer than two samples would not do this. However, what happens when fewer than two samples are taken per cycle is not necessarily intuitive. It turns out that the sinusoid masquerades as one of another frequency.

Consider the situation when the frequency f of a continuous-time sinusoid is 7,560 Hz. Figure 11.3 shows 4.5 msec of the continuous-time waveform, together with samples taken at 8 kHz. Notice that the samples trace out another sinusoid. We can determine the frequency of that sinusoid with the help of figure 11.2, which suggests that the perceived pitch will be $8000 - 7560 = 440$ Hz (the slope of the perceived pitch line is -1 in this region). Indeed, if we listen to the sampled sinusoid, it will be an A-440.

Recall that a cosine can be given as a sum of complex exponentials with frequencies that are negatives of one another. Recall further that a complex exponential with frequency f is indistinguishable from one with frequency $f + Nf_s$, for any integer N. A variant of figure 11.2 that leverages this representation is given in figure 11.4.

In figure 11.4, as we sweep the frequency of the continuous-time signal from 0 to 8 kHz, we move from left to right in the figure. The sinusoid consists not only of the rising frequency shown by the dotted line in figure 11.2, but also of a corresponding falling (negative) frequency as shown in figure

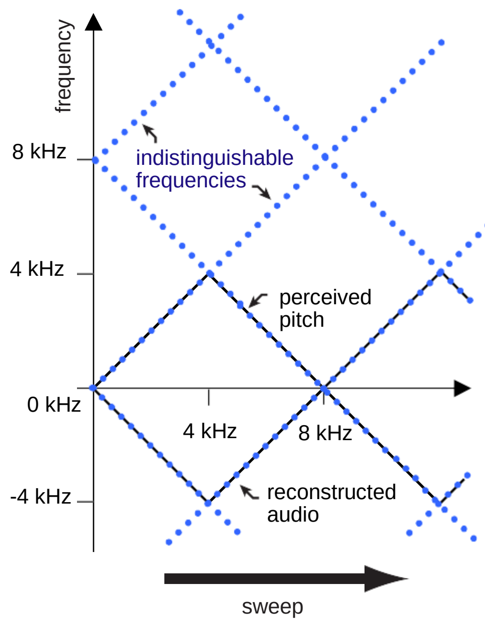


Figure 11.4: As the frequency of a continuous signal increases beyond the Nyquist frequency, the perceived pitch starts to drop because the frequency of the reconstructed continuous-time audio signal stays in the range $-f_s/2$ to $f_s/2$.

11.4. Moreover, these two frequencies are indistinguishable, after sampling, from frequencies that are 8 kHz higher or lower, also shown by dotted lines in figure 11.4.

When the discrete-time signal is converted to a continuous-time audio signal, the hardware performing this conversion can choose any matching pair of positive and negative frequencies. By far the most common choice is to select the matching pair with lowest frequency, shown in figure 11.4 by the solid lines behind dotted lines. These result in a sinusoid with frequency between 0 and the Nyquist frequency, $f_s/2$. This is why the perceived pitch falls after sweeping past $f_s/2 = 4$ kHz.

Recall that the frequency-domain representation (i.e. the DTFT) of a discrete-time signal is periodic with period 2π radians/sample. That is, if X is a DTFT, then

$$\forall\, \omega \in Reals, \quad X(\omega) = X(\omega + 2\pi).$$

In radians per second, it is periodic with period $2\pi f_s$. In Hertz, it is periodic with period f_s, the sampling frequency. Thus, in figure 11.4, the dotted lines represent this periodicity. This periodicity is another way of stating that frequencies separated by f_s are indistinguishable.

Avoiding aliasing ambiguities

Figure 11.4 suggests that even though a discrete-time signal has ambiguous frequency content, it is possible to construct a uniquely defined continuous-time signal from the discrete-time waveform by choosing the one unique frequency for each component that is closest to zero. This will always result in a reconstructed signal that contains only frequencies between zero and the Nyquist frequency.

Correspondingly, this suggests that when sampling a continuous-time signal, if that signal contains only frequencies below the Nyquist frequency, then this reconstruction strategy will perfectly recover the signal. This is an intuitive statement of the **Nyquist-Shannon sampling theorem**.

If a continuous-time signal contains only frequencies below the Nyquist frequency $f_s/2$, then it can be perfectly reconstructed from samples taken at sampling frequency f_s. This suggests that prior to sampling, it is reasonable to filter a signal to remove components with frequencies above $f_s/2$. A filter that realizes this is called an **anti-aliasing filter**.

> **Example 11.3:** In the telephone network, speech is sampled at 8000 samples per second before being digitized. Prior to this sampling, the speech signal is lowpass filtered to remove frequency components above 4000 Hz. This lowpass filtered speech can then be perfectly reconstructed at the far end of the telephone connection, which receives a stream of samples at 8000 sample per second.

Before probing this further, let us examine in more detail what we mean by reconstruction.

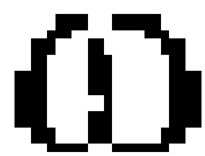
Reconstruction

Consider a system that constructs a continuous-time signal x from a discrete-time signal y,

$$DiscToCont_T : DiscSignals \to ContSignals.$$

Probing further: Anti-Aliasing for Fonts

When rendering characters on a computer screen, it is common to use **anti-aliasing** to make the characters look better. Consider the two figures below:



At the left is an image of the Greek letter omega. At the right is the result of sampling that rendition by taking only one pixel out of every 100 pixels in the original (every 10-th pixel horizontally and vertically), and then rescaling the image so it has the same size as the one on the left. The original image is discrete, and the resulting image is a smaller discrete image (this process is known as **subsampling**). Rendered with normal-sized pixels the character on the right looks like this:

To the discerning eye, this can be improved considerably. The problem is that the character at the upper left above has hard edges, and hence high (spatial) frequencies. Those high frequencies result in aliasing distortion when subsampling. To improve the result, we first lowpass filter the character (blurring it), and then subsample, as shown below:

The result looks better to the discerning eye:

Figure 11.5: Discrete to continuous converter.

This is illustrated in figure 11.5. Systems that carry out such ‘discrete-to-continuous’ conversion can be realized in any number of ways. Some common examples are illustrated in figure 11.6, and defined below:

- **zero-order hold**: This means simply that the value of the each sample $y(n)$ is held constant for duration T, so that $x(t) = y(n)$ for the time interval from $t = nT$ to $t = (n + 1)T$, as illustrated in figure 11.6(b). Let this system be denoted

$$ZeroOrderHold_T : DiscSignals \to ContSignals.$$

- **linear interpolation**: Intuitively, this means simply that we connect the dots with straight lines. Specifically, in the time interval from $t = nT$ to $t = (n+1)T$, $x(t)$ has values that vary along a straight line from $y(n)$ to $y(n+1)$, as illustrated in figure 11.6(c). Linear interpolation is sometimes called **first-order hold**. Let this system be denoted

$$LinearInterpolator_T : DiscSignals \to ContSignals.$$

- **ideal interpolation**: It is not yet clear what this should mean, but intuitively, it should result in a smooth curve that passes through the samples, as illustrated in figure 11.6(d). We will give a precise meaning below. Let this system be denoted

$$IdealInterpolator_T : DiscSignals \to ContSignals.$$

A model for reconstruction

A convenient mathematical model for reconstruction divides the reconstruction process into a cascade of two systems, as shown in figure 11.7. Thus

$$x = S(ImpulseGen_T(y)),$$

where S is an LTI system to be determined. The first of these two subsystems,

$$ImpulseGen_T : DiscSignals \to ContSignals,$$

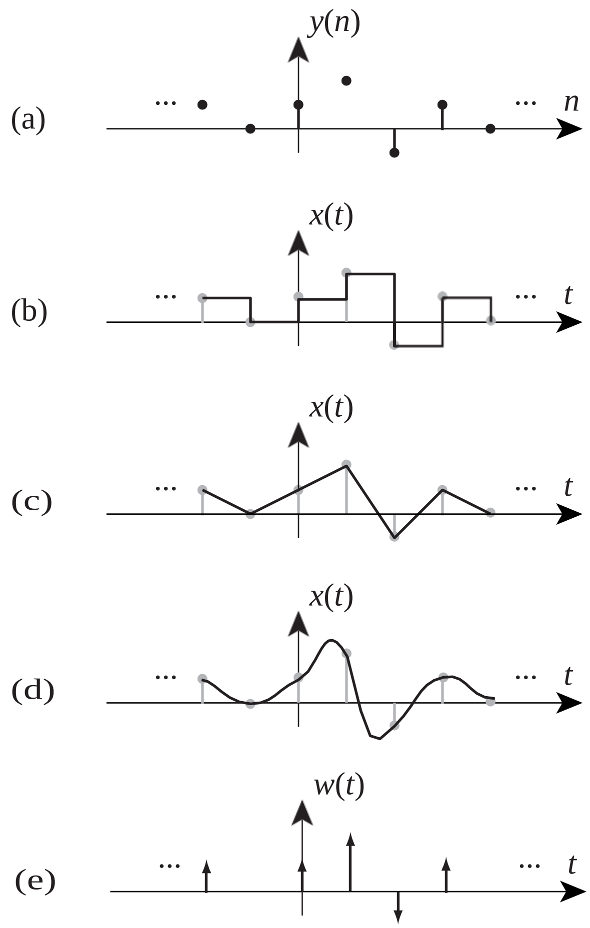


Figure 11.6: A discrete-time signal (a), a continuous-time reconstruction using zero-order hold (b), a reconstruction using linear interpolation (c), a reconstruction using ideal interpolation (d), and a reconstruction using weighted Dirac delta functions (e).

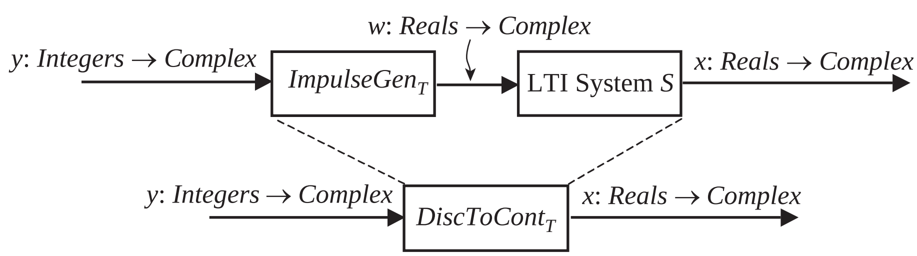


Figure 11.7: A model for reconstruction divides it into two stages.

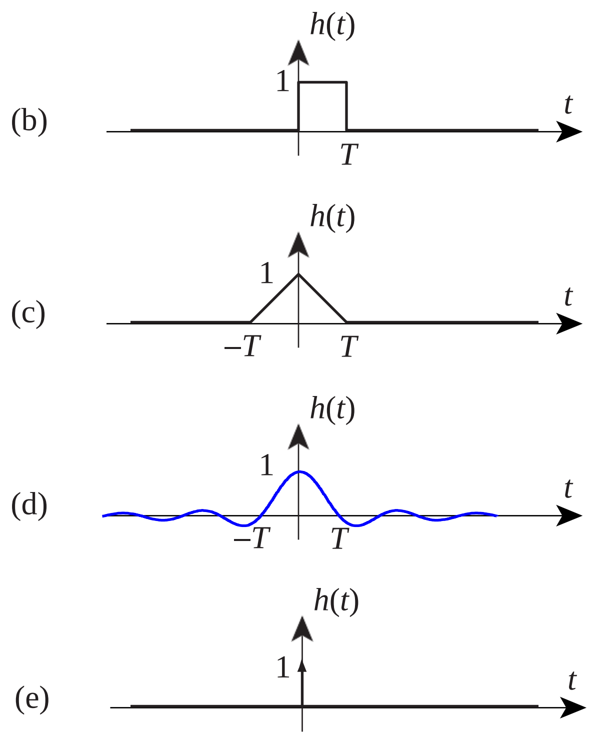


Figure 11.8: The impulse responses for the LTI system S in figure 11.7 that yield the interpolation methods in figure 11.6(b-e).

constructs a continuous-time signal, where for all $t \in Reals$,

$$w(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t - kT).$$

This is a continuous-time signal that at each sampling instant kT produces a Dirac delta function with weight equal to the sample value, $y(k)$. This signal is illustrated in figure 11.6(e). It is a mathematical abstraction, since everyday engineering systems do not exhibit the singularity-like behavior of the Dirac delta function. Nonetheless, it is a useful mathematical abstraction.

The second system in figure 11.7, S, is a continuous-time LTI filter with an impulse response that determines the interpolation method. The impulse responses that yield the interpolation methods in figure 11.6(b-e) are shown in figure 11.8(b-e). If

$$\forall\, t \in Reals, \quad h(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & otherwise \end{cases}$$

then the interpolation method is zero-order hold. If

$$\forall\, t \in Reals, \quad h(t) = \begin{cases} 1 + t/T & -T < t < 0 \\ 1 - t/T & 0 \le t < T \\ 0 & otherwise \end{cases}$$

then the interpolation method is linear. If the impulse response is

$$\forall\, t \in Reals, \quad h(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

then the interpolation method is ideal. The above impulse response is called a **sinc function**, and its Fourier transform, from table 10.4, is given by

$$\forall\, \omega \in Reals, \quad X(\omega) = \begin{cases} T & \text{if } |\omega| \leq \pi/T \\ 0 & \text{otherwise} \end{cases}$$

Notice that the Fourier transform is zero at all frequencies above π/T radians/second, or $f_s/2$ Hz, the Nyquist frequency. It is this characteristic that makes it ideal. It precisely performs the strategy illustrated in figure 11.4, where among all indistinguishable frequencies we select the ones between $-f_s/2$ and $f_s/2$.

If we let $Sinc_T$ denote the LTI system S when the impulse response is a sinc function, then

$$IdealInterpolator_T = Sinc_T \circ ImpulseGen_T.$$

In practice, ideal interpolation is difficult to accomplish. From the expression for the sinc function we can understand why. First, this impulse response is not causal. Second, it is infinite in extent. More importantly, its magnitude decreases rather slowly as t increases or decreases (proportional to $1/t$ only). Thus, truncating it at finite length leads to substantial errors.

If the impulse response of S is

$$h(t) = \delta(t),$$

where δ is the Dirac delta function, then the system S is a pass-through system, and the reconstruction consists of weighted delta functions.

The Nyquist-Shannon sampling theorem

We can now give a precise statement of the **Nyquist-Shannon sampling theorem**:

> If x is a continuous-time signal with Fourier transform X and if $X(\omega)$ is zero outside the range $-\pi/T < \omega < \pi/T$ radians/second, then
>
> $$x = IdealInterpolator_T(Sampler_T(x)).$$

We can state this theorem slightly differently. Suppose x is a continuous-time signal with no frequency larger than some f_0 Hertz. Then x can be recovered from its samples if $f_0 < f_s/2$, the Nyquist frequency.

Probing further: Sampling

We can construct a mathematical model for sampling by using Dirac delta functions. Define a pulse stream by

$$\forall\, t \in Reals, \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Consider a continuous-time signal x that we wish to sample with sampling period T. That is, we define $y(n) = x(nT)$. Construct first an intermediate continuous-time signal $w(t) = x(t)p(t)$. We can show that the CTFT of w is equal to the DTFT of y. This gives us a way to relate the CTFT of x to the DTFT of its samples y. Recall that multiplication in the time domain results in convolution in the frequency domain (see table 10.9), so

$$W(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) P(\omega - \Omega) d\Omega.$$

It can be shown (see box on page 386 that the CTFT of $p(t)$ is

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}),$$

so

$$\begin{aligned} W(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \Omega - k\frac{2\pi}{T}) d\Omega \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(\Omega) \delta(\omega - \Omega - k\frac{2\pi}{T}) d\Omega \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T}) \end{aligned}$$

where the last equality follows from the sifting property (9.11). The next step is to show that

$$Y(\omega) = W(\omega/T).$$

We leave this as an exercise. From this, the basic Nyquist-Shannon result follows,

$$Y(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right).$$

This relates the CTFT X of the signal being sampled x to the DTFT Y of the discrete-time result y.

Probing further: Impulse Trains

Consider a signal p consisting of periodically repeated Dirac delta functions with period T,

$$\forall\, t \in Reals, \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

This signal has the Fourier series expansion

$$\forall\, t \in Reals, \quad p(t) = \sum_{m=-\infty}^{\infty} \frac{1}{T} e^{i\omega_0 mt},$$

where the fundamental frequency is $\omega_0 = 2\pi/T$. This can be verified by applying the formula from table 10.5. That formula, however, gives an integration range of 0 to the period, which in this case is T. This integral covers one period of the periodic signal, but starts and ends on a delta function in p. To avoid the resultant mathematical subtleties, we can integrate from $-T/2$ to $T/2$, getting Fourier series coefficients

$$\forall\, m \in Integers, \quad P_m = \frac{1}{T} \int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} \delta(t - kT) \right) e^{i\omega_0 mt} dt.$$

The integral is now over a range that includes only one of the delta functions. The kernel of the integral is zero except when $t = 0$, so by the sifting rule, the integral evaluates to 1. Thus, all Fourier series coefficients are $P_m = 1/T$. Using the relationship between the Fourier series and the Fourier Transform of a periodic signal (from section 10.6.3), we can write the continuous-time Fourier transform of p as

$$\forall\, \omega \in Reals, \quad P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T} k\right).$$

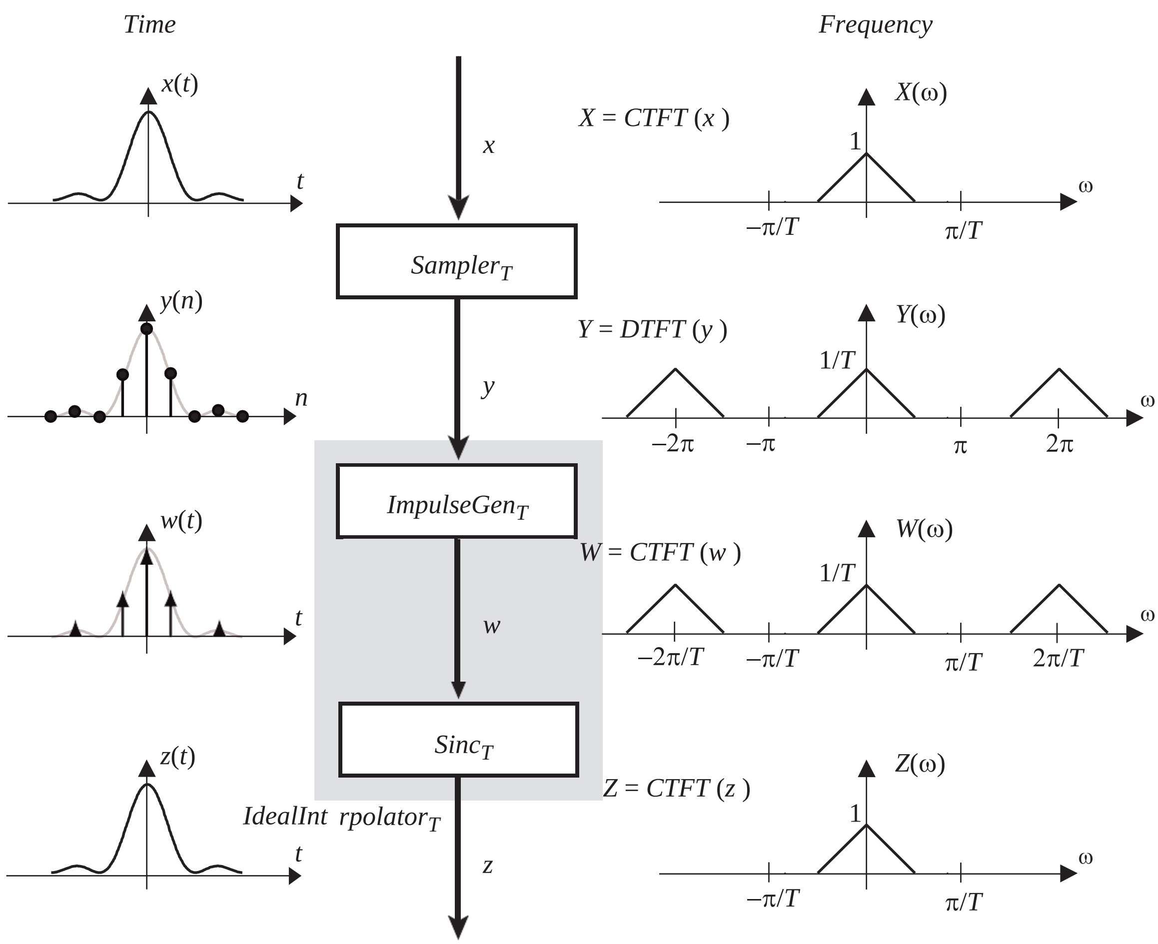


Figure 11.9: Steps in the justification of the Nyquist-Shannon sampling theorem.

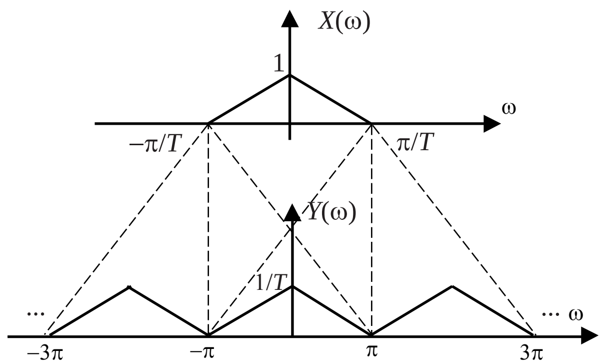


Figure 11.10: Relationship between the CTFT of a continuous-time signal and the DTFT of its discrete-time samples. The DTFT is the sum of the CTFT and its copies shifted by multiples of $2\pi/T$, the sampling frequency in radians per second. The frequency axis is also normalized.

A formal proof of this theorem involves some technical difficulties (it was first given by Claude Shannon of Bell Labs in the late 1940s). But we can get the idea from the following three-step argument (see figure 11.9).

Step 1. Let x be a continuous-time signal with Fourier transform X. At this point we do not require that $X(\omega)$ be zero outside the range $-\pi/T < \omega < \pi/T$. We sample x with sampling interval T to get the discrete-time signal

$$y = Sampler_T(x).$$

It can be shown (see box on page 385) that the DTFT of y is related to the CTFT of x by

$$\boxed{Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right).}$$

This important relation says that the DTFT Y of y is the sum of the CTFT X with copies of it shifted by multiples of $2\pi/T$. Also, the frequency axis is normalized by dividing ω by T . There are two cases to consider, depending on whether the shifted copies overlap.

First, if $X(\omega) = 0$ outside the range $-\pi/T < \omega < \pi/T$, then the copies will not overlap, and in the range $-\pi < \omega < \pi$,

$$Y(\omega) = \frac{1}{T}X\left(\frac{\omega}{T}\right). \tag{11.5}$$

In this range of frequencies, Y has the same shape as X , scaled by $1/T$. This relationship between X and Y is illustrated in figure 11.10, where X is drawn with a triangular shape.

In the second case, illustrated in figure 11.11, X does have non-zero frequency components higher than π/T. Notice that in the sampled signal, the frequencies in the vicinity of π are distorted by the overlapping of frequency components above and below π/T in the original signal. This distortion is called **aliasing distortion**.

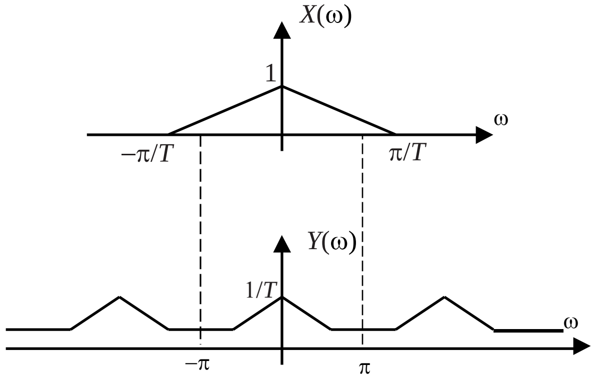


Figure 11.11: Relationship between the CTFT of a continuous-time signal and the DTFT of its discrete-time samples when the continuous-time signal has a broad enough bandwidth to introduce aliasing distortion.

We continue with the remaining steps, following the signals in figure 11.9.

Step 2. Let w be the signal produced by the impulse generator,

$$\forall\, t \in Reals, \quad w(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - nT).$$

The Fourier Transform of w is $W(\omega) = Y(\omega T)$ (see box on page 385).

Step 3. Let z be the output of the $IdealInterpolator_T$. Its Fourier transform is simply

$$\begin{aligned} Z(\omega) &= W(\omega)S(\omega) \\ &= Y(\omega T)S(\omega), \end{aligned}$$

where $S(\omega)$ is the frequency response of the reconstruction filter $IdealInterpolator_T$. As seen in exercise 21 of chapter 10,

$$S(\omega) = \begin{cases} T & -\pi/T < \omega < \pi/T \\ 0 & \text{otherwise} \end{cases} \tag{11.6}$$

Substituting for S and Y, we get

$$\begin{aligned} Z(\omega) &= \begin{cases} TY(\omega T) & -\pi/T < \omega < \pi/T \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \sum_{k=-\infty}^{\infty} X(\omega - 2\pi k/T) & -\pi/T < \omega < \pi/T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If $X(\omega)$ is zero for $|\omega|$ larger than the Nyquist frequency π/T, then we conclude that

$$\forall\, \omega \in Reals, \quad Z(\omega) = X(\omega).$$

That is, w is identical to x. This proves the Nyquist-Shannon result.

However, if $X(\omega)$ does have non-zero values for some $|\omega|$ larger than the Nyquist frequency, then z will be different from x, as illustrated in figure 11.11.

Summary

The acts of sampling and reconstructing a continuous-time signal bridge the continuous-time world with the discrete computational world. The periodicity of frequencies in the discrete world implies that for each discrete-time sinusoidal signal, there are multiple corresponding discrete-time frequencies. These frequencies are aliases of one another. When a signal is sampled, these frequencies become indistinguishable, and aliasing distortion may result. The Nyquist-Shannon sampling theorem gives a simple condition under which aliasing distortion is avoided. Specifically, if the signal contains no sinusoidal components with frequencies higher than half the sampling frequency, then there will be no aliasing distortion. Half the sampling frequency is called the Nyquist frequency because of this key result.

Exercises

Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

1. **E** Consider the continuous-time signal

$$x(t) = \cos(10\pi t) + \cos(20\pi t) + \cos(30\pi t).$$

   (a) Find the fundamental frequency. Give the units.

   (b) Find the Fourier series coefficients $A_0, A_1, \cdots$ and $\phi_1, \phi_2, \cdots$.

   (c) Let y be the result of sampling this signal with sampling frequency 10 Hz. Find the fundamental frequency for y, and give the units.

   (d) For the same y, find the discrete-time Fourier series coefficients, $A_0, A_1, \cdots$ and $\phi_1, \cdots$.

   (e) Find

$$w = IdealInterpolator_T(Sampler_T(x))$$

   for $T = 0.1$ seconds.

   (f) Is there any aliasing distortion caused by sampling at 10 Hz? If there is, describe the aliasing distortion in words.

   (g) Give the smallest sampling frequency that avoids aliasing distortion.

2. **E** Verify that $Sampler_T$ defined by (11.1) and (11.2) is linear but not time invariant.

3. **E** A real-valued sinusoidal signal with a negative frequency is always exactly equal to another sinusoid with positive frequency. Consider a real-valued sinusoid with a negative frequency -440 Hz,

$$y(n) = \cos(-2\pi 440 nT + \phi).$$

Find a positive frequency f and phase θ such that

$$y(n) = \cos(2\pi f nT + \theta).$$

4. **T** Consider a continuous-time signal x where for all $t \in Reals$,

$$x(t) = \sum_{k=-\infty}^{\infty} r(t-k).$$

where

$$r(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ 0 & otherwise \end{cases}.$$

   (a) Is $x(t)$ periodic? If so, what is the period?
   (b) Suppose that $T = 1$. Give a simple expression for $y = Sampler_T(x)$.
   (c) Suppose that $T = 0.5$. Give a simple expression for $y = Sampler_T(x)$ and $z = IdealInterpolator_T(Sampler_T(x$
   (d) Find an upper bound for T (in seconds) such that $x = IdealInterpolator_T(Sampler_T(x))$, or argue that no value of T makes this assertion true.

5. **T** Consider a continuous-time signal x with the following finite Fourier series expansion,

$$\forall t \in Reals, \quad x(t) = \sum_{k=0}^{4} \cos(k\omega_0 t)$$

where $\omega_0 = \pi/4$ radians/second.

   (a) Give an upper bound on T (in seconds) such that $x = IdealInterpolator_T(Sampler_T(x))$.
   (b) Suppose that $T = 4$ seconds. Give a simple expression for $y = Sampler_T(x)$.
   (c) For the same $T = 4$ seconds, give a simple expression for

$$w = IdealInterpolator_T(Sampler_T(x)).$$

6. **T** Consider a continuous-time audio signal x with CTFT shown in figure 11.12. Note that it contains no frequencies beyond 10 kHz. Suppose it is sampled at 40 kHz to yield a signal that we will call x_{40}. Let X_{40} be the DTFT of x_{40}.

   (a) Sketch $|X_{40}(\omega)|$ and carefully mark the magnitudes and frequencies.
   (b) Suppose x is sampled at 20,000 samples/second. Let x_{20} be the resulting sampled signal and X_{20} its DTFT. Sketch and compare x_{20} and x_{40}.
   (c) Now suppose x is sampled at 15,000 samples/second. Let x_{15} be the resulting sampled signal and X_{15} its DTFT. Sketch and compare X_{20} and X_{15}. Make sure that your sketch shows aliasing distortion.

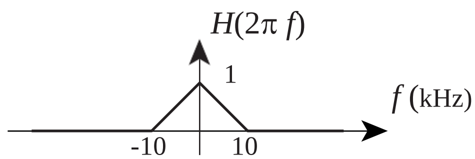


Figure 11.12: CTFT of an audio signal considered in exercise 6.

7. **C** Consider two continuous-time sinusoidal signals given by

$$x_1(t) = \cos(\omega_1 t)$$

$$x_2(t) = \cos(\omega_2 t),$$

with frequencies ω_1 and ω_2 radians/second such that

$$0 \le \omega_1 \le \pi/T \quad \text{and} \quad 0 \le \omega_2 \le \pi/T.$$

Show that if $\omega_1 \neq \omega_2$ then

$$Sampler_T(x_1) \neq Sampler_T(x_2).$$

I.e., the two distinct sinusoids cannot be aliases of one another if they both have frequencies below the Nyquist frequency. **Hint**: Try evaluating the sampled signals at $n = 1$.